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AN ALGORITHM FOR SOLVING AN INTEGER LINEAR PROGRAMMING PROBLEM
WITH A FIXED NUMBER OF VARIABLES
IN POLYNOMIAL TIME

by

A. RATTANAVINYOO

A thesis submitted to the Faculty
of Graduate Studies and Research in
partial fulfillment of the requirements
for the degree of Master of Science

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December 1984
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Faculty of Graduate Studies and
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ABSTRACT

This thesis begins by giving background information on Integer Linear Programming (ILP) and Lattices including the methods of solving an ILP problem such as Rounding Method, Branch and Bound Method, and Cutting Plane Method.

An ILP problem is an Np Complete problem. There is no known algorithm to solve an ILP problem which has a running time bounded by a polynomial function of the input length. H.W. Lenstra has defined an algorithm to solve an ILP problem for a fixed number of variables. The complexity of his algorithm is bounded by a polynomial function of the input length.

The details of Lenstra's algorithm are described in this thesis. His algorithm based on Geometry of Numbers [Lattices]. The idea of the algorithm is to transform the ILP problem into an equivalent one and also transform the integer points into lattice points. This transformed problem has the following additional property: either the existence of the integer solution in the feasible region of the ILP problem is obvious or it is known that the last coordinate of the integer solution belongs to an interval whose length is bounded by a constant only depending on the number of variables. An important part of Lenstra's algorithm is a Basis Reduction process for lattices which is of independent interest.
ACKNOWLEDGEMENT

I would like to take this opportunity to thank Professor B. Mortimer for his encouragement and all the valuable advice during the span of this thesis. Also, I would like to thank my family and my friends for their support.
satisfies the integer requirements (1.2), it is an optimum solution of the ILP and we are done. On the other hand, if one or more of the \( x_j, \ (1 \leq j \leq n), \) are non-integer, \( x \) is not feasible for the ILP.

One may be tempted to conclude that the optimal solution of the ILP can be obtained by rounding off \( x \) to a neighbouring integer solution. This rounding off method may be a practical method to use, and may give an excellent approximation to the optimum integer solution. This is clearly the case when the solution values are large numbers. Suppose, for example, an integer programming problem in three variables attacked via the simplex algorithm, yields the following solution:

\[
x_1 = 326.2, \quad x_2 = 1074.9, \quad x_3 = 195.5.
\]

It seems a fair assumption that the value of the objective function in an optimum integer programming solution to this problem would not be significantly different from that produced by the following solution, obtained by rounding:

\[
x_1 = 326, \quad x_2 = 1075, \quad x_3 = 196.
\]

Truncating the fractional part of non-integer answers may also work. Consider the programming problem in two variables given in Figure 1.1.

If both \( x_1, x_2 \) are to be integers, the optimum solution is clearly \( x_1 = x_2 = 1, \) and \( f \) is maximized at \( f_1 \) (as shown in Figure 1.1). By truncating the fractional part of the non-integer solution, we obtain an optimum integer solution.

However, rounding and truncating will sometimes get us into trouble. For example, if we change the second constraint in problem 1.1, we can obtain such a problem as in Problem 1.2.

As a linear programming problem, the optimum solution is \( x_1 = 1.87, x_2 = 1.87 \). Clearly from Figure 1.2, if the problem were altered by the added requirement that both \( x_1 \) and \( x_2 \) must be
INTRODUCTION

Integer linear programming (ILP) is the following optimization problem.

\[
\text{maximize } cx
\]

subject to \( Ax \leq b \)

\( x \geq 0 \), integers,

where \( A \) is an \((m \times n)\)-matrix with integral coefficients, \( b \) is an \( m \)-row vector with integral coefficients, \( m, n \) are positive integers, and \( c \) is a cost function. The problem is to find an \( n \)-vector \( x \) with integral coordinates \( x_j \), \((1 \leq j \leq n)\), satisfying the inequalities \( Ax \leq b \) and maximizing \( cx \). Since the ILP problem is a \( \text{NP} \)-Complete problem, there is no known algorithm for finding a solution vector \( x \) of this problem which has a running time that is bounded by a polynomial function of the length of the data. This length may (details in Chapter 2) be defined to be \( nm \log(n + 2) \) where \( n \) denotes the maximum of the absolute values of the coefficients of \( A \) and \( b \).

Consider any ILP problem. In the case \( n = 1 \), it is trivial to design a polynomial algorithm for the solution of the problem. For \( n = 2 \), Hirschberg and Wong [8] and Kannan [9] have given polynomial algorithms in special cases which we shall discuss later in Chapter 4. In this thesis, we will present the algorithm defined by H.W. Lenstra. He shows that for any fixed value of \( n \) there exists a polynomial time algorithm for the solution of the ILP problem. The running time of the algorithm is bounded by a polynomial function of the length of the input.

In fact, his algorithm solves the linear inequality problem instead of the ILP problem. Since solving an ILP problem is no more difficult than solving a system of linear inequalities. We can develop a solution
to the ILP problem from this algorithm in the following way. First, solve the ILP problem by relaxing the integer constraints (this can be done by using simplex method or an ellipsoid method). Secondly, apply this algorithm to the ILP problem, the result of this algorithm being to decide whether there is at least one integer point in the feasible region. If there is one then we can reduce the problem to a new ILP problem with a smaller feasible region. A new ILP problem is constructed from the original constraints and a new constraint which is \( cx \geq cx_2 \) where \( x_2 = (x_1 + x_3)/2 \). and \( x_1 \) is the optimal solution for the ILP problem by relaxing the integer constraints, \( x_2 \) is any integer point in the feasible region of the original ILP found by applying this algorithm, and \( c \) is a cost function. This is a binary search algorithm, repeat step 2 and solve for an integer solution of the new problem (if there exists one); otherwise, construct a new problem with the original constraints and two more constraints which are \( cx \geq cx_2 \) and \( cx \leq cx_3 \), and solve for an integer point of this problem. Continue recursively, until we conclude that there are no more integer points in the new feasible region then we terminate with the last integer point as the optimal solution for the ILP problem. If there does not exist any integer point in the original problem (no such \( x_2 \) ) then terminate with the conclusion that the ILP problem has no solution.

The complexity of solving an ILP problem by the above method is also bounded by a polynomial in the length of the input. To see this; in step 1, it is clear that it can be done within a time bounded by a polynomial in the length of the input. For step 2, the number of times that we search for integer points between \( x_1 \) and \( x_3 \) is at most \( \log(x_1 - x_2) \). Thus there are at most \( \log(x_1 - x_3) \) iterations of the algorithm to find any integer point in the feasible region. This algorithm can be proved (later in this thesis) to have a running time that is bounded by a polynomial function of the length of the input.

Our algorithm is described in Section 3.1. We want to find an integer point satisfying the system.
We will transform the problem into a new problem by a non-singular linear transform \( r \), and transform \( Z^n \) to a lattice \( \mathcal{L} \) by \( r \). Now we have an equivalent problem such that \( A' \mathbf{y} \leq b \), where \( A' = Ar^{-1} \), solve for \( \mathbf{y} \in \mathcal{L} \), if there exists one. In this new formulation, the algorithm proceeds as follows: either the existence of a vector \( \mathbf{y} \in \mathcal{L} \) is obvious, which means that there exists a vector \( \mathbf{y} = y_1 \mathbf{u}_1 + \cdots + y_n \mathbf{u}_n \), where \( y_1, \ldots, y_n \) are integers, and \( \mathbf{u}_1, \ldots, \mathbf{u}_n \) is a basis for \( \mathcal{L} \), satisfying \( A' \mathbf{y} \leq b \), or it is known that the last coordinate of any such \( \mathbf{y} \) belongs to an interval whose length is bounded by a constant only depending on \( n \). In the latter case, the problem is reduced to a bound number of lower-dimensional problems.

Before the transformation can be made, we need, first, to remodel the convex set \( \{ \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq b \} \) and construct the linear transformation \( r \); this algorithm is given in Step 1 and Step 2 of the algorithm in Section 2.1. Secondly, we will change a given basis for a lattice \( \mathcal{L} \) into one satisfying the inequality \( \prod_{i=1}^n |b_i| \leq c_2 d(\mathcal{L}) \), where \( b_1, \ldots, b_n \) is a basis for \( \mathcal{L} \), \( c_2 \) is a constant only depending on \( n \), and \( d(\mathcal{L}) \) is the determinant of \( \mathcal{L} \). This latter algorithm is known as the basis-reduction process for \( n \)-dimensional lattices. It is described in Chapter 3.

We let \( K \) be the convex set defined by the system \( A\mathbf{x} \leq b \), so that \( K = \{ \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq b \} \). Our problem is to find an integer vector in \( K \), that is, an integer solution to \( A\mathbf{x} \leq b \). We apply an appropriate \( r \) to \( K \) such that \( rK \) contains a regular simplex (this is a better shape, because \( rK \) is equilateral), and also apply an appropriate \( r \) to \( Z^n \) forming a lattice \( \mathcal{L} \). Now, we can decide whether \( rK \) contains a lattice point, i.e. if \( rK \cap \mathcal{L} \neq \emptyset \). To do so, we need to get a basis of \( \mathcal{L} \) such that \( \prod_{i=1}^n |b_i| \leq c_2 d(\mathcal{L}) \), as mentioned before. With all these preparations, we get a new problem with additional properties, and we can decide whether \( K \) contains an integer point. At the end of Chapter 2.
we will show that this can be done in a time bounded by a polynomial in the length of the input for a fixed value of $n$.

As P. Van Emde Boas observed, it follows directly from Lenstra's main result that the ILP problem with a fixed value of $m$ is also polynomial solvable, the detail is in Section 4.1.
Chapter 1

BASIC CONCEPTS

In this Chapter, we would like to introduce some concepts and some definitions used throughout this thesis, and also mention integer linear programming.

1.1 Integer linear programming (ILP) problems

An integer linear programming problem is:

\[
\begin{align*}
\text{maximize} & \quad cx \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]  

(1.1)

where \( A \) is a matrix of order \( m \times n \). The problem obtained by relaxing the integer requirement (1.2) is known as the associated linear programming problems (LP). The following are methods of solving ILP problems.

1.1.1 Rounding off and truncating method

Solve the associated LP, by the simplex method for example. If its optimum solution,

\[ x = (x_1, x_2, \ldots, x_n), \]
satisfies the integer requirements (1.2), it is an optimum solution of the ILP and we are done. On the other hand, if one or more of the \( x_j \), \((1 \leq j \leq n)\), are non-integer, \( \bar{x} \) is not feasible for the ILP.

One may be tempted to conclude that the optimal solution of the ILP can be obtained by rounding off \( \bar{x} \) to a neighbouring integer solution. This rounding off method may be a practical method to use, and may give an excellent approximation to the optimum integer solution. This is clearly the case when the solution values are large numbers. Suppose, for example, an integer programming problem in three variables, attacked via the simplex algorithm, yields the following solution:

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If both \( x_1, x_2 \) are to be integers, the optimum solution is clearly \( x_1 = x_2 = 1 \), and \( f \) is maximized at \( f_3 \) (as shown in Figure 1.1). By truncating the fractional part of the non-integer solution, we obtain an optimum integer solution.

However, rounding and truncating will sometimes get us into trouble. For example, if we change the second constraint in problem 1.1, we can obtain such a problem as in Problem 1.2.

As a linear programming problem, the optimum solution is \( x_1 = 1.87, x_2 = 1.87 \). Clearly from Figure 1.2, if the problem were altered by the added requirement that both \( x_1 \) and \( x_2 \) must be
Problem 1.1:

maximize \( f = x_1 + 3x_2 \)

subject to \( x_1 \leq 1.87 \)
\( x_2 \leq 1.87 \).

Figure 1-1:

Integers, truncating would get us into trouble, the optimum integer solution is not that corresponding to the point \((1,1)\) but rather to the point \((2,1)\).

Efficiency of the rounding off and truncating method

This rounding off is a procedure that can be used in some simple integer programming models, and its constraints should be in the model so that it is easy to decide whether feasibility is preserved by rounding off a non-integer value of a variable to the next larger integer or the nearest integer lower than
Problem 1.2

maximize \( f = x_1 + 3x_2 \)

subject to \( x_2 \leq 1.87 \)

\[ 60x_1 + 28x_2 \leq 165. \]

Figure 1-2: Constraint 2 \[ 60x_1 + 28x_2 \leq 165 \]

This method can also give us trouble if a feasible neighbouring integer point of a solution is far away from the optimum integer solution. This method is not suitable for any ILP problems whose values of an optimum solution of the problem are likely to be small.
1.1.2 Branch and bound method

This algorithm solves an ILP problem, by locating the nearest feasible integer point, which maximizes the objective function of the problem. This method is an application of the branch and bound method (see [3]). Problem 1.3 is an example problem of the branch and bound method.

**Problem 1.3**

\[
\text{maximize } f = x_1 + 3x_2 \\
\text{subject to } 22x_1 + 34x_2 \leq 105 \\
x_2 \leq 1.87 \\
x_1, x_2 \geq 0, \text{ integers.}
\]

![Figure 1-3](image)
We begin by solving the linear programming problem that results if constraint 3 is eliminated. Using the simplex algorithm of linear programming, we obtain \( x_1 = 1.87, x_2 = 1.87 \), and the upper bound (UB) of this problem is \( x_1 + 3x_2 = (1.87) + 3(1.87) = 7.5 \). We associate this solution with node 1 in Figure 1.4.

We now introduce constraint 3, and partition the set of further constrained problems into those in which \( x_1 \leq 1 \) on the one hand and those in which \( x_2 \geq 2 \) on the other hand; that is, we investigate values of \( x_1 \) on either side of \( x_1 = 1.87 \). We represent these two classes of problems as nodes 2 and 3 respectively, in Figure 1.4.

We now solve these two problems using the simplex algorithm, and produce the solutions associated with nodes 2 and 3 in Figure 1.4. The resulting objective function values constitute upper bounds on all further constrained problems in their respective classes, as shown in Figure 1.4; and the LP's below any node have a feasible set contained in the feasible set of the root. That is to say, examining nodes 2, 3, we conclude that the objective function of nodes 2 and 3 cannot be greater than 7.5. For node 2, we also conclude that the objective of any linear program in which \( x_1 \leq 1 \) and in which \( x_2 \) is not only constrained to be greater than or equal to zero, but is further constrained to be an integer, cannot be greater than 6.61. Similarly, referring to node 3, we see that the objective function of any linear programming problem in which \( x_1 \geq 2 \) and in which \( x_2 \) is only constrained to be greater than or equal to zero, but is furthered constrained to be an integer, cannot be greater than 7.4.

We repeat the process for nodes 4, 5, ..., 8. From Figure 1.4, the problem represented by node 5 is not feasible, then we terminate from that branch. Finally, we get an optimum solution as shown in node 8, such that \( x_1 = 3, x_2 = 1 \) and the upper bound is 6.
1.1.3 Cutting planes method

Let \( z \) be an optimum solution of the associated LP and suppose \( z \) is non-integer. A cutting plane
is defined to be a hyperplane that strictly separates $x$ from the set of integer feasible solutions of this problem, that is, it is a hyperplane with $x$ on one side of it ($x$ cannot lie on the hyperplane) and all the integer feasible solutions of the problem either on the hyperplane or on the other side of it. Figure 1.5 contains an illustration of a cutting plane. The associated LP has four constraints and $x$ is an optimum solution for it. The convex hull of all the integer feasible solutions is the polytope outlined with dashed lines. The vector $x$ is on one side of the cutting plane and all the integer feasible solutions are on its other side.

The cutting plane method proceeds as follows:

a) Let the current problem be the current associated LP. Solve the current problem. It can be solved by the simplex method, starting with an optimum solution of the previous problem.
If the current problem is infeasible, the original integer problem is also infeasible. Terminate.

If the optimum solution of the current problem satisfies the integer requirements, it is an optimum solution of the original integer program. Terminate.

If the optimum solution of the current problem violates some integer restrictions, go to b)

b) Use the cut generation method, from a) consider a row appearing in the final simplex tableau. A rule for choosing a row from the final simplex tableau is to choose a row with the largest fractional part in the non-integer solution value of the variables. We can generate the new constraint (a cutting plane) from this row. Replace all the coefficients in the constraint equation under consideration by the smallest non-negative numbers that are congruent to these coefficients, and then state that the resulting expression must be greater than (less than) or equal to the fractional part of the constant on the right hand side of the equal sign. Two numbers are congruent if their difference is an integer, zero being considered an integer. Thus, $27/14$ is congruent to $13/14$, $-3/5$ is congruent to $2/5$.

Add this new constraint to the current problem. Go to a).

The method is repeated until it terminates. We will only consider cutting plane methods satisfying the following property: the method must terminate either with the conclusion that the problem is infeasible or with an optimum integer solution in a finite number of steps.

An example of ILP problem using a cutting plane method

**Problem 1.4**

maximize $3x_1 + 3x_2$

subject to $2x_1 + 4x_2 \leq 10$

$5x_1 + 4x_2 \leq 12$
$x_1, x_2 \geq 0$, integers.

Conical form:

Maximize $3x_1 + 3x_2$

Subject to

$2x_1 + 4x_2 + s_1 = 10$

$5x_1 + 4x_2 + s_2 = 12$

$x_1, x_2, s_1, s_2 \geq 0$, integers.

Invoking the simplex procedure, we produce the final tableau in the table 1.1.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>5/12</td>
<td>-1/6</td>
<td>13/6</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-1/3</td>
<td>1/3</td>
<td>2/3</td>
<td></td>
</tr>
</tbody>
</table>

Table 1-1: Solution $x_1 = \frac{2}{3}, x_2 = 2\frac{1}{3}$

Choose the second constraint to produce a new constraint (a cutting plane), since $x_1 = 2/3$ is the variable with the largest fractional part in the non-integer solution value. From the second constraint equation we produce the cut

$$0x_1 + 0x_2 + \frac{2}{3}s_1 + \frac{1}{3}s_2 \leq \frac{2}{3}$$

$$2s_1 + s_2 \leq 2$$

Solving for $s_1$ and $s_2$ in the original constraint equations and substituting, we transform this to

$$3x_1 + 4x_2 \leq 10$$

(cut 1 in Figure 1.6).
Add the new constraint (new cut) to the current problem, and solve via the simplex procedure. Since the first constraint is no longer binding (from Figure 1.6), we drop it. We produce the final tableau in Table 1.2.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-3/8</td>
<td>5/8</td>
</tr>
</tbody>
</table>

Table 1.2:
From the second constraint equation we produce the cut

\[
\frac{5}{8} s_3 + \frac{5}{8} s_4 \geq \frac{3}{4} \text{ or } 5x_1 + 5x_2 \leq 13 \quad \text{(cut 2 in Figure 1.6)}
\]

Adding the new cut to the current problem and solving via the simplex procedure, we produce the final tableau in Table 1.3.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_5$</th>
<th>$s_6$</th>
<th>$s_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-4/5$</td>
<td>8/5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$-8/5$</td>
<td>6/5</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1.3:

From the first constraint equation, we produce the cut

\[ s_7 \geq 3 \text{ or } x_1 + x_2 \leq 2 \quad \text{(cut 3 in Figure 1.6).} \]

Repeating the process, finally we get an optimal solution of $x_1 = 2, x_2 = 0$.

**Efficiency of cutting plane method**

Even though it can be proven that the cutting plane algorithm solves integer programs in a finite number of steps, this finite number can be very large. For simple problems, this algorithm requires a very large number of steps before termination, and in most other practical problems, this method is not suitable.
1.2 Some definitions

Let \( a_1, \ldots, a_n \) be linearly independent real vectors in \( n \)-dimensional real Euclidean space, so that the only set of numbers \( t_1, \ldots, t_n \) for which \( t_1 a_1 + \cdots + t_n a_n = 0 \) is \( t_1 = \cdots = t_n = 0 \). The set of all points

\[
X = u_1 a_1 + \cdots + u_n a_n,
\]

with integral \( u_1, \ldots, u_n \) is called the lattice with basis \( a_1, \ldots, a_n \). Since \( a_1, \ldots, a_n \) are linearly independent, the expression of any vector \( X \) in (1.3) with \( u_1, \ldots, u_n \) is unique.

The basis is not uniquely determined by the lattice. Let \( a'_i \) be the points

\[
a'_i = \sum_j v_{ij} a_j.
\]

where \( v_{ij} \) are any integers with

\[
|\det(v_{ij})| = \pm 1,
\]

then

\[
a_i = \sum_j w_{ij} a'_j,
\]

with integral \( w_{ij} \). It follows easily that the set of points (1.3) is precisely the set of points

\[
u_1 a'_1 + \cdots + u_n a'_n,
\]

where \( u'_1, \ldots, u'_n \) are integers, that is \( a_1, \ldots, a_n \) and \( a'_1, \ldots, a'_n \) are bases of the same lattice. On substituting (1.4) in (1.6) and making use of the linear independence of the \( a_i \), we have:
\[
\sum_{i,j} w_{ij}v_{ji} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}
\]

Hence, \(\text{det}(w_{ij})\text{det}(v_{ji}) = 1\), note that \(\text{det}(w_{ij}) = \text{determinant of } [w_{ij}]_{1 \leq i,j \leq n}\) so each of \(\text{det}(w_{ij})\) and \(\text{det}(v_{ji})\) must be \(\pm 1\), that is (1.5) holds as required.

Determinant of a lattice, \(d(\Lambda)\)

Let \(a_1, \ldots, a_n\) and \(b_1, \ldots, b_n\) be bases of the same lattice, so that they are related by (1.4), (1.5).

Then we have

\[
\text{det}(b_1, \ldots, b_n) = \text{det}(v_{ij})\text{det}(a_1, \ldots, a_n) = \pm \text{det}(a_1, \ldots, a_n).
\]

Hence, if we define \(d(\Lambda)\), the determinant of the lattice \(\Lambda\), as \(|\text{det}(a_1, \ldots, a_n)|\) then \(d(\Lambda)\) is independent of the particular choice of basis, and \(d(\Lambda) > 0\), since \(a_1, \ldots, a_n\) are linearly independent.

Lattices under transformation

Let the linear transformation \(X = \alpha x\) be given by

\[
X_i = \sum_{1 \leq j \leq n} a_{ij}x_j \quad (1 \leq i \leq n) \tag{1.7}
\]

where \(X = (X_1, \ldots, X_n), x = (x_1, \ldots, x_n)\), are corresponding points in the transformation and \(a_{ij}\) are real numbers such that \(\text{det}(\alpha) = \text{det}(\alpha_{ij}) \neq 0\). This is a non-singular linear transformation.

Let \(\Lambda\) be a lattice and denote by \(\alpha \Lambda\) the set of points \(\alpha x, x \in \Lambda\). If \(b_1, \ldots, b_n\) is a basis for \(\Lambda\), then the general point \(b = u_1b_1 + \cdots + u_nb_n\), where \(u_1, \ldots, u_n\) are integers of \(\Lambda\) has the transform

\[
\frac{ab}{\alpha} = \alpha(u_1b_1 + \cdots + u_nb_n) = u_1\alpha b_1 + \cdots + u_n\alpha b_n.
\]

Hence, \(\alpha \Lambda\) is a lattice with basis \(\alpha b_1, \ldots, \alpha b_n\), and

\[
d(\alpha \Lambda) = |\text{det}(\alpha b_1, \ldots, \alpha b_n)| = |\text{det}(\alpha)|d(\Lambda) = |\text{det}(\alpha)|d(\Lambda).
\]
Inequality of Hadamard

Lemma 1 Let \( a_1, \ldots, a_n \) be \( n \)-dimensional vectors. Then

\[
|\det(a_1, \ldots, a_n)| \leq |a_1| \ldots |a_n|.
\]

Proof

If the determinant is 0, there is nothing to prove. Suppose that \( a_1, \ldots, a_n \) are linearly independent.

We construct a sequence of vectors \( c_j \) \((1 \leq j \leq n)\) such that

\[
c_i c_j = 0 \quad (i \neq j).
\]

This is just a scalar product of two vectors, and

\[
a_i = t_{i1} c_1 + \cdots + t_{i,j} c_j + c_i,
\]

for some real numbers \( t_{i,1} \), hence

\[
c_i = a_i.
\]

and define \( c_i \) recursively.

\[
c_i = a_i - \sum_{j \neq i} (a_i c_j) |c_j|^{-2} c_j.
\]

By (1.8), (1.9) we have

\[
|a_i|^2 = a_1 a_1
\]

\[
= t_{11}^2 |c_1|^2 + \cdots + t_{n-1}^2 |c_{n-1}|^2 + |c_i|^2
\]

\[
\geq |c_i|^2.
\]

Since,

\[
c_1 = \sigma_1 \quad \text{and}
\]

\[
\sigma_1 |\sigma_1| = \sigma_1^2.
\]
\[ c_2 = a_2 - (a_2 c_1)|c_1|^{-2} c_1 = a_2 - k_{11} a_1, \]

where \( k_{11} = (a_2 c_1)|c_1|^{-2} \),

\[ c_3 = a_3 - (k_{21} a_1 + k_{22} c_2) = a_3 - (k_{21} + k_{22} k_{11}) a_1 - k_{22} a_2. \]

where \( k_{21} = (a_3 c_1)|c_1|^{-2} \) and \( k_{22} = (a_3 a_2)|c_1|^{-2} \). From above, we can see that \( c_i \) is a linear combination of \( a_i \). Hence

\[ \text{det}(a_1, \ldots, a_n) = \text{det}(c_1, \ldots, c_n). \quad (1.11) \]

Regarding the \( c_1, \ldots, c_n \) in \( \text{det}(c_1, \ldots, c_n) \) first as rows and then as columns and multiplying the two determinants together, we have

\[ \left\{ \text{det}(c_1, \ldots, c_n) \right\}^2 = \text{det}\{c_i, c_j\} = \prod_{i=1}^{n} |c_i|^2. \quad (1.12) \]

From (1.10), (1.11), (1.12) we have

\[ |\text{det}(a_1, \ldots, a_n)| \leq |a_1| \ldots |a_n|. \quad (1.13) \]

This inequality represents the fact that the volume of a parallelepiped is at most the product of the length of the sides.

Convex hull

The convex hull is the natural boundary of a point set. The convex hull of a set of points in the plane is defined to be the smallest convex polygon containing them all. A convex polygon is a polygon with the property that the line segment connecting any two points inside the polygon must itself lie inside the polygon. A fundamental property of the convex hull is that any line outside the hull, when moved in any direction towards the hull, either hits the hull at one of its vertex points, or hits the hull at one of its edges.
A hyperplane in $\mathbb{R}^n$ is the set of all points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, satisfying a single linear equation $a_1 x_1 + \cdots + a_n x_n = b$, where $a_1, \ldots, a_n$ are given numbers and at least one of the $a_i$ is non-zero, that is $(a_1, \ldots, a_n) \neq 0$.

\textbf{N-Simplex}

In $N$ dimensions, a simplex is the set of all points $x = x_0 v^0 + \cdots + x_n v^n$ such that $x_0, \ldots, x_n \geq 0$ and $x_0 + \cdots + x_n = 1$; the $n+1$ fixed vectors $v^0, v^1, \ldots, v^n$ are the vertices of the simplex. The numbers $x_0, \ldots, x_n$ are by definition the barycentric coordinates of the vector $x$. The simplex is nondegenerate if the $n$ vertices $v^1 - v^0, v^2 - v^0, \ldots, v^n - v^0$ are linearly independent and the barycenter $b$ of an $n$-dimensional simplex is the center of gravity and is such that $b = \frac{1}{n+1} (v^0 + v^1 + \cdots + v^n)$.

\textbf{The class $\mathcal{P}$}

The class $\mathcal{P}$ is the set of all problems which can be solved by deterministic algorithms in polynomial time. By a deterministic algorithm, we mean that the algorithm has only one thing it can do next.

\textbf{The class $\mathcal{NP}$}

The class $\mathcal{NP}$ is the set of all problems that can be solved by non-deterministic algorithms in polynomial time. By a non-deterministic algorithm, we mean that the algorithm when faced with a choice of several options, has the power to guess the right one.

$\mathcal{NP}$-Complete problem is the term used to describe problems that are the hardest ones in $\mathcal{NP}$ in the sense that no $\mathcal{NP}$-Complete problem can be solved by any known polynomial time algorithm, and if there is a polynomial time algorithm for any $\mathcal{NP}$-Complete problem, then there are polynomial time algorithms for all $\mathcal{NP}$-Complete problems.
1.3 Some concepts

The following concepts are used throughout this thesis. We shall prove them in this section, and later in the thesis, we will refer to them. The first concept concerns the nature of the convex set of the system $Ax \leq b$. We will show that we may assume that this convex set is bounded and has a positive volume; by positive volume, we mean that a convex hull with vertices $v_0, v_1, ..., v_n$ has a dimension of full dimension. The next concept concerns a regular simplex $(S)$, an equilateral simplex. There exist two closed balls with center $p$ (barycenter of $S$) such that $B(p, r) \subset S \subset B(p, R)$, where $r, R$ are the radii, and $R/r$ is bounded by some constant value. As the last concept, we shall show that, we can find an upper bound of the distance between any vector $x \in \mathbb{R}^n$ and a vector $y \in$ lattice.

1.3.1 An assumption for Convex Hulls

Let $K$ be the closed convex set, $K = \{x \in \mathbb{R}^n : Ax \leq b\}$, where $A$ is an $(m \times n)$-matrix, $b \in \mathbb{Z}^n$. We would like to ensure

a) $K$ is bounded

b) $K$ has a positive volume (full dimension).

To show a), we shall invoke a result of Von zur Gathen and Sieveking, [4], stating that $K \cap \mathbb{Z}^n \neq \emptyset$ if and only if $K \cap \mathbb{Z}^n$ contains a vector whose coefficients are bounded by $(n + 1)n^{-\frac{n+1}{2}}2^{-n^2}$ in absolute value, where $a$ denotes the maximum of the absolute values of the coefficients of $A$ and $b$. Adding these inequalities to the system makes $K$ bounded.

If $K$ is bounded, $K$ is also a convex hull of the set of its vertices. A vertex is obtained by replacing a linearly independent inequalities in the system $Ax \leq b$ by equalities, and testing whether the unique
vector satisfying these equalities also satisfies the other inequalities; if it does, it is a vertex of $K$.

To justify b), let $v_0, v_1, \ldots, v_M$ be the vertices for the system $Ax \leq b$. Hence, $K$ contains

$v_0, v_1, \ldots, v_M$ and $M > 0$ if $K$ is not empty. Let $d$ be the dimension of the linear subspace $\mathcal{V}$ of $\mathbb{R}^n$ generated by the vectors $v_j - v_0$, $1 \leq j \leq M$. If $d = n$, where $n$ is the number of variables in the system $Ax \leq b$, then there are $n + 1$ vertices among $v_0, v_1, \ldots, v_M$ whose convex hull is an $n$-simplex of positive volume. For example

The system of

\[
x_1 + 2x_2 + x_3 \leq 4
\]

\[
x_1 + 2x_2 + 2x_3 \leq 6
\]

\[
x_1, x_2, x_3 \geq 0.
\]

A convex hull: $v_0 = 0, 0, 0$

$v_1 = 0, 0, 3$

$v_2 = 4, 0, 0$

$v_3 = 0, 2, 0$

Since $n=3$ and $d=3$, a convex hull is a 3-simplex of positive volume.

We will now prove that even if $d < n$ we may assume that the convex hull, $K$, has a positive volume. We need to show that the problem can be transformed into an integer programming problem with only $d$ variables. Choose, for each $j$, an integer multiple $w_j$ of $v_j - v_0$ such that $w_j \in \mathbb{Z}^n$. and let $W$ be the $(n \times M)$-matrix whose columns are $w_j$. There exists an integral $(n \times n)$-matrix $U$ with $\text{det}(U) = \pm 1$, such that

\[
UW = [k_{ij}]_{1 \leq i \leq n, 1 \leq j \leq M}.
\]

\[
\begin{cases}
  k_{ij} = 0 & \text{if } i > j \text{ or } i > d \\
  k_{ii} \neq 0 & \text{if } 1 \leq i \leq d.
\end{cases}
\]
Such a matrix $U$ can be found in polynomial time by repeated application of the Euclidean algorithm.

An example of finding a matrix $U$

Let $W = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 7 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

Step 1 Find a matrix $U_{31}$ with integral coefficient and $\det U_{31} = \pm 1$ such that

$$U_{31}W = [k_{ij}]_{1 \leq i, j \leq 3} : \text{with}$$

$$k_{11} = \gcd (w_{11}, w_{21}) \land k_{31} = 0.$$

Then take

$$U_{31} = \begin{bmatrix} s_1 & t_1 & 0 \\ -\frac{s_{21}}{k_{11}} & \frac{t_{21}}{k_{11}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that

$$\begin{bmatrix} s_1 & t_1 & 0 \\ -\frac{s_{21}}{k_{11}} & \frac{t_{21}}{k_{11}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix},$$

where $s_1, t_1$ are chosen such that these three equations are satisfied.

$$\left( \frac{w_{11}}{k_{11}} \right) s_1 - \left( -\frac{w_{21}}{k_{11}} \right) t_1 = 1 \quad \text{(This is possible by the Euclidean algorithm).}$$

$$w_{12}s_1 + w_{22}t_1 = k_{12}$$

$$\{ \text{k}_{12}, k_{13} \text{are integers.} \}$$

$$w_{13}s_1 + w_{23}t_1 = k_{13}$$

In our case,

$$U_{31} = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 7 & 3 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & k_{12} & k_{13} \\ 0 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$
Let $s_1 = -1$, $t_1 = 1$. Hence,

$$U_{s_1} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad K_1 = U_{s_1}w = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$  

**Step 2** Repeat Step 1, find a matrix $U_{s_1}$ with integral coefficients and $\det U_{s_1} = \pm 1$ such that $U_{s_1}K_1 = [k'_{ij}]_{i,j \leq 3}$ with

$$k'_{11} = \gcd(k_{11}, k_{31}),$$

$$k'_{21} = 0.$$

Then take

$$U_{s_1} = \begin{bmatrix} s_2 & 0 & t_2 \\ 0 & 1 & 0 \\ \frac{-s_2}{k'_{11}} & 0 & \frac{-t_2}{k'_{11}} \end{bmatrix}, \quad \text{with} \quad \begin{pmatrix} k_{11} \\ k'_{11} \end{pmatrix} s_2 - \begin{pmatrix} -k_{31} \\ k'_{31} \end{pmatrix} t_2 = 1.$$  

Select $s_2, t_2$ (the same way as $s_1, t_1$ were selected), let $s_2 = -1, t_2 = 4$, then

$$U_{s_1} = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}, \quad \text{and} \quad K_2 = U_{s_1}K_1 = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}.$$  

**Step 3** Find a matrix $U_{s_2}$ with integral coefficients and $\det U_{s_2} = \pm 1$ such that $U_{s_2}K_2 = [k''_{ij}]_{i,j \leq 3}$ with

$$k''_{32} = 0,$$

$$k''_{22} = \gcd(k'_{22}, k''_{22}).$$

Using the same process as in Step 1 and Step 2, we can obtain

$$U_{s_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix}, \quad \text{and} \quad K_3 = U_{s_2}K_2 = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 7 \\ 0 & 0 & 11 \end{bmatrix}.$$  

We stop at this step, since $K_3 = [k''_{ij}]_{i,j \leq 3}$ with

$$\begin{cases} k''_{ij} = 0 & \text{if } i > j \\ k''_{ii} \neq 0 & \text{if } 1 \leq i \leq 3 \end{cases}.$$
Step 4 We can obtain $U$ as $U = U_{21}U_{21}U_{11}$; and $\det U = \det U_{21} \det U_{21} \det U_{11} = \pm 1$. In this example

$$U = \begin{bmatrix}
1 & -1 & 4 \\
-4 & 3 & -6 \\
5 & -4 & 9 \\
\end{bmatrix}$$

In general, the formula for such a matrix $U$, such that $U_i^\text{T}W = K_i$, where $K_i$ is a matrix with element $k_{ij} = 0$ for $i > j$, is

$$u_{ij} = 0, \quad u_{ij} = t$$

$$u_{ij} = -\frac{w_{ij}}{g}, \quad u_{ii} = \frac{w_{jj}}{g}; \text{ where } g = \gcd(w_{ij}, w_{ii})$$

$$u_{ef} = 0, \quad u_{ee} = 1; \text{ for all } (e, f) \neq (i, j),$$

and

$$\left(\frac{w_{jj}}{g}\right) + \left(\frac{w_{ij}}{g}\right)t = 1$$

Hence, $U$ is a unimodular matrix. Denote by $u_1, \ldots, u_n$ the columns of the integral matrix $U^{-1}$, then $U^{-1}$ is also a unimodular matrix with $\det U^{-1} = \pm 1$. These $u_1, u_2, \ldots, u_n$ form a basis of $\mathbb{Z}^n$ and $Z = \sum_{j=1}^n Z u_j$. The linear subspace $V$ of $\mathbb{Z}^n$ generated by the vectors $v_j - v_0, 1 \leq j \leq M$, or in other word $K \subset V + v_0$; and let $W$ be the $(n \times M)$ matrix whose $j$-column is an integer multiple of $v_j - v_0$, the columns of $W = U^{-1}[k_{ij}]$, so (1.13) implies that

$$V = \sum_{j=1}^d \mathbb{Z} u_j.$$ \hfill (1.14)

Let $v_0 = \sum_{j=1}^n r_j u_j$, ($r_j \in \mathbb{R}, 1 \leq j \leq n$). Now suppose that $x \in K \cap Z^n$, then $x = \sum_{j=1}^n y_j u_j$, with $y_j \in Z, 1 \leq j \leq n$,

$$x - v_0 = (y_1 - r_1)u_1 + \cdots + (y_d - r_d)u_d + (y_{d+1} - r_{d+1})u_{d+1} + \cdots + (y_n - d_n)u_n.$$ \hfill (1.15)

Suppose $x - v_0 \in V$, then by (1.14), (1.15) with $y_j = r_j, d < j \leq n$. If at least one of the $r_{d+1}, \ldots, r_n$ is not an integer, then this implies that $x - v_0 \notin V$, since $y_j \in Z (1 \leq j \leq n)$; hence $K \cap Z^n = \emptyset$ (no
such $x$). On the other hand, $r_{d+1}, \ldots, r_n$ are all integer, so that the problem can be transformed to an integer programming with only $d$ variables $y_1, \ldots, y_d$, as required. Hence, we can therefore assume that we have a case of full dimension.

1.3.2 To find a radius of the Euclidean ball for a regular simplex

Lenstra derives the way of finding a radius for a Euclidean ball of a regular simplex such that:

Lemma 2 By $C(u_0, \ldots, u_n)$, we denote the convex hull of $u_0, \ldots, u_n$, and by $\text{vol}$ we denote the volume of the convex hull. Let $z_0, z_1, \ldots, z_n \in \mathbb{R}^n$ be the vertices of a regular $n$-simplex $S$, with barycenter $p = \frac{1}{n+1} \sum_{i=0}^n z_i$. Let $T$ be the set of all $x \in \mathbb{R}^n$ for which

$$\text{vol}(C(x, z_0, z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)) \leq \text{vol}S,$$

for $i = 0, 1, \ldots, n$. Then there are positive real numbers, $r$ and $R$ such that

$$B(p,r) \subset S \subset T \subset B(p,R)$$

and

$$\left(\frac{R}{r}\right)^2 = \left\{ \begin{array}{ll}
n^2 + 2n^3 & \text{if } n \text{ is even} \\
n^2 + n^3 - n & \text{if } n \text{ is odd.}
\end{array} \right.$$  

where $B(p,r)$ is a Euclidean closed ball in $\mathbb{R}^n$ with center $p$ and radius $r$; that is

$$B(p,r) = \{ x \in \mathbb{R}^n : |x - p| \leq r \},$$

where $p \in \mathbb{R}^n$, $r \in \mathbb{R} \geq 0$, $|\cdot|$ denotes the Euclidean length in $\mathbb{R}^n$.

Proof We can identify $\mathbb{R}^n$ with the hyperplane $\{ (r_j)_{j=0}^n \in \mathbb{R}^{n+1} : \sum_{j=0}^n r_j = 1 \}$ in $\mathbb{R}^{n+1}$ such that $z_0, z_1, \ldots, z_n$ is the standard basis of $\mathbb{R}^{n+1}$. Then we have $p = \left( \frac{1}{n+1}, \ldots, \frac{1}{n+1} \right)$, and

$$T = \{ (r_j)_{j=0}^n \in \mathbb{R}^{n+1} : |r_j| \leq 1 \text{ for } 0 \leq j \leq n \text{ and } \sum_{j=0}^n r_j = 1 \}$$
It is easily seen that \( T \) is the convex hull of the set of points obtained by permuting the coordinates of the point \( \sum_{j=0}^{\frac{n}{2}} z_j - \sum_{j=\frac{n}{2}+1}^{n} z_j \), where \( n = 2k \) if \( n \) is even, or \( n = 2k + 1 \) if \( n \) is odd.

It follows that \( T \subset B(p, R) \), where \( R \) is the distance of \( p \) to the above point.

\[
R^3 = \begin{cases} 
\frac{n^3 + 2n}{n+1} & \text{if } n \text{ is even} \\
\frac{n^3 + n^3 - n}{n+1} & \text{if } n \text{ is odd.}
\end{cases}
\]

Further, \( B(p, r) \subset S \), where \( r \) is the distance of \( p \) to \( \left( 0, \frac{1}{n}, \ldots, \frac{1}{n} \right) \) the center of one of the faces of \( S \):

\[
r^3 = \frac{1}{n(n+1)}.
\]

Since, trivially \( S \subset T \) and

\[
B(p, r) \subset S \subset T \subset B(p, R)
\]

\[
\left( \frac{R}{r} \right)^3 = \begin{cases} 
\frac{n^3 + 2n}{n^3 + n^3 - n} & \text{if } n \text{ is even} \\
\frac{n^3 + 2n}{n^3 + n^3 - n} & \text{if } n \text{ is odd.}
\end{cases}
\]

### 1.3.3 An upper bound of a nearest lattice point from any vector

Let \( \mathcal{L} = \sum_{i=1}^{n} Z h_i = \{ \sum_{i=1}^{n} m_i h_i : m_i \in Z, (1 \leq i \leq n) \} \). Then \( \mathcal{L} \) is a lattice in \( \mathbb{R}^n \) and \( b_1, b_2, \ldots, b_n \) is a basis for \( \mathbb{R}^n \).

**Lemma 3** Let \( b_1, b_2, \ldots, b_n \) be any basis for \( \mathcal{L} \). For any vector \( x \in \mathbb{R}^n \), there exists a vector \( y \in \mathcal{L} \) such that

\[
|x - y|^2 \leq \frac{1}{4} (|b_1|^2 + \ldots + |b_n|^2), \tag{1.18}
\]

and furthermore if we number \( b_i \) such that

\[
|b_n| = \max\{|b_i| : 1 \leq i \leq n\},
\]
then (1.18) implies that
\[
\forall x \in \mathbb{R}^n : \exists y \in \mathcal{L} : |x - y| \leq \frac{1}{2} \sqrt{n}|b_n|.
\] (1.19)

**Proof** The proof is by induction on \( n \), the case \( n = 1 \) (or \( n = 0 \)) being obvious since vector \( x \) lies between \( y_1, y_2 \) where \( x \in \mathbb{R}^1 \) and \( y_1, y_2 \in \mathcal{L} \). For example, let \( b_1 = 1 \), and
\[ x = a(1) ; a \in \mathbb{R} . \]
\[ y_1 = b(1) ; b \in \mathbb{Z} \text{ and } b < a . \]
\[ y_2 = c(1) ; c \in \mathbb{Z} \text{ and } a < c . \]
Hence, the distance from \( x \) to either \( y_1 \) or \( y_2 \) is at most \( \frac{1}{2}|b_1| \). \( |x - y_i|^2 \leq \frac{1}{4}|b_1|^2 \).

Assume that it is true for a lattice of dimension \( n - 1 \). Then
\[
\forall x \in \mathbb{R}^{n-1} : \exists y \in \mathcal{L}' : |x - y|^2 \leq \frac{1}{4}(|b_1|^2 + \ldots + |b_{n-1}|^2).
\]
where \( \mathcal{L}' = \sum_{i=1}^{n-1} \mathbb{Z}b_i \), this is a lattice in the \((n-1)\) dimensional hyperplane \( H = \sum_{i=1}^{n-1} \mathbb{R}b_i \). Denote by \( h \) the distance of \( b_n \) to \( H \). Clearly we have
\[
h \leq |b_n| . \] (1.20)

Now, to prove (1.18), let \( x \in \mathbb{R}^n \). We wish to change \( x \) by an element of \( \mathcal{L} \) such that its length becomes small. First subtract an integral multiple of \( b_n \) from \( x \) such that its distance to \( H \) becomes \( \leq \frac{1}{2}h \). Write \( x = x_1 + x_2 \), with \( x_1 \in H \) and \( x_2 \) perpendicular to \( H \), then \( |x_2| \leq \frac{1}{2}h \leq \frac{1}{2}|b_n| \). By the induction hypothesis, we can change \( x_1 \) by an element of \( \mathcal{L}' \) and achieve that \( |x_1|^2 \leq \frac{1}{4}(|b_1|^2 + \ldots + |b_{n-1}|^2) \), since \( x_1, x_2 \) are perpendicular to each other, \( |x_1|^2 + |x_2|^2 \leq \frac{1}{4}(|b_1|^2 + \ldots + |b_{n-1}|^2) + \frac{1}{4}(|b_n|^2) \) or \( |x|^2 \leq \frac{1}{4}(|b_1|^2 + \ldots + |b_n|^2) \). This proves (1.18). If we number the \( b_i \) such that \( |b_n| = \max \{|b_i| : 1 \leq i \leq n\} \), then \( (|b_1|^2 + \ldots + |b_n|^2) \leq n(|b_n|^2) \), from (1.18) we implies that
\[
|x - y|^2 \leq \frac{1}{4}n|b_n|^2 \quad \text{or} \quad |x - y| \leq \frac{1}{2}\sqrt{n}|b_n| .
\]
This proves (1.19).
Chapter 2

An algorithm for solving the ILP problem within a polynomial time

This chapter describes the algorithm to transform an ILP problem into an equivalent one with an additional property as described in the Introduction: either the existence of a vector $x \in \mathbb{Z}^n$ satisfying $Ax \leq b$ is obvious, or it is known that the last coordinate of any such $x$ belongs to an interval whose length is bounded by a constant only depending on $n$. This algorithm is only concerned with the constraints of an ILP problem, and the algorithm will terminate if there exists an integer point in the feasible region. We shall describe the algorithm in Section 2.1. In order to understand the algorithm clearly, we shall present an example of this algorithm in Section 2.2.

2.1 An algorithm to transform an ILP problems

Let an ILP problem have constraints $Ax \leq b$, where $A$ is a matrix with integer coefficients of order $m \times n$, and $b$ is a column vector of order $m$. Let $K$ be the convex hull of the system $Ax \leq b$.

Throughout this algorithm, we solve any LP problem by the Simplex method. In the worst case, the complexity of this method is not a polynomial running time. In practice, it is a very practical method and its worst case is seldom occurs. The algorithm is as follows:

Step 1 Find a Simplex in $K$

Maximize an arbitrary function on $K$; by using the simplex method, maximize an arbitrary function
subject to $Ax \leq b$, $x \geq 0$. Note that $x$ is not necessarily an integer vector. One finds a vertex $v_0$ of $K$ unless $K$ is empty. Now to construct, by induction, vertices $v_1, \ldots, v_d$ of $K$ such that the vectors $v_j - v_0$, $1 \leq j \leq d$, are linearly independent. Suppose that vertices $v_0, \ldots, v_d$ for $d < n$ have been constructed for which $v_1 - v_0, \ldots, v_d - v_0$ are linearly independent. By induction, we mean to maximize or minimize several linear functions on $K$ that are constant on $\{v_0, \ldots, v_d\}$. For example, we can find the equation of the plane passing through $\{v_0, \ldots, v_d\}$, then maximize this function on $K$. We might find $v_{d+1}$ (if it exists), check whether $v_j - v_0$, $1 \leq j \leq d + 1$, are linearly independent. We consider 2 cases.

Case 1. We can find a vertex $v_{d+1}$ such that $v_1 - v_0, \ldots, v_d - v_0$ are linearly independent, and if

\[(d + 1) = n,\] then we stop. This is the case of full dimension.

Case 2. We discover that $K \subset v_0 + V$, where $V = \sum_{j=1}^{d} \mathbb{R}(v_j - v_0)$. In this case, as described in Subsection 1.5.1 of Chapter 1, we can change the original problem with $n$ variables into one with only $d$ variables. This mean that there doesn't exist $v_{d+1}$ such that $v_1 - v_0, \ldots, v_d - v_0, v_{d+1} - v_0$ are linearly independent. Now, $W$ which is a matrix of an integral multiple of $v_j - v_0$ where $1 \leq j \leq M$, as in Subsection 1.5.1, is an $(n \times d)$-matrix rather than $(n \times M)$. This is a problem with $d$ dimension.

Note: If the LP problem is infeasible in this step, terminate from the algorithm, and conclude that the ILP problem also infeasible.

Step 2. Find a large Simplex inside $K$.

Select a positive real number $\epsilon$. For some $i$, find a vertex $v$ of $K$ such that

$$\text{vol}(C(v, v_0, \ldots, v_i, \ldots, v_n)) \geq (1 + \epsilon) \text{vol}(C(v_0, v_1, \ldots, v_n)).$$

where $C(v_0, \ldots, v_n)$ denotes the convex hull, and vol the volume. If such a $v$ exists, it can be found
by solving two linear programming problems for fixed $i$. The two linear programming problems can be
constructed by maximizing and minimizing the linear function $\text{vol}(C(v_0, v_1, \ldots, v_{i-1}, v_{i+1}, v_n))$, subject
to the original constraints. Now, replace $v_i$ by $v$ (if such exists) and repeat the process for each $i$ until
$i = n$, keeping $v$ fixed. If there doesn't exist such $v$ then the convex hull is unchanged. With every
step $\text{vol}(C(v_0, \ldots, v_n))$ increases by a factor of at least $(1 + \epsilon)$. On the other hand, it is bounded by
$\text{vol}(K)$. After this step we have a simplex in $K$ spanned by $v_0, \ldots, v_n$ such that

$$\text{vol}(C(x, v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)) < (1 + \epsilon)\text{vol}(C(v_0, \ldots, v_n));$$

for all $x \in K$ and for all $i \in \{0, 1, \ldots, n\}$. \hfill (2.1)

**Step 3 Mapping the Simplex to a regular Simplex**

The convex hull $C(v_0, \ldots, v_n)$ in Step 2 forms an $n$-dimensional simplex $D$ of $K$. Choose a linear
transformation $r$ of $\mathbb{R}^n$ such that the vectors $z_j = r(v_j)$ for $0 \leq j \leq n$, span a regular $n$-dimensional
simplex $S$. Hence, $D$ is transformed by $r$ into $S$; this is an affine transformation. From Step 2, $\text{vol}(D)$
is bounded by $\text{vol}(K)$, $\text{vol}(D) \leq \text{vol}(K)$; and

$$\text{vol}(S) = |\det r| \cdot \text{vol}(D) \leq |\det r| \cdot \text{vol}(K) = \text{vol}(rK).$$

then $S \subset rK$. If we apply $r$ to (2.1):

$$\text{vol}(C(y, z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)) < (1 + \epsilon) \text{vol}(rD) = (1 + \epsilon) \text{vol}(S);$$

where $y = r(x)$ for all $x \in K$ and for all $i \in \{0, 1, \ldots, n\}$. \hfill (2.2)

Let

$$T_\epsilon = \{ z \in \mathbb{R}_n : \forall i \in \{0, 1, \ldots, n\} ;$$
\[ \text{vol}(C(z, z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)) < (1 + \epsilon) \text{vol}(S). \] (2.3)

Since \( K \in \mathbb{R}^n \) and \( r \) is a linear transformation, then from (2.2), (2.3) : \( rK \subset T_\epsilon \). Recall the concept from Subsection (1.3.2) of Chapter 1: \( S \subset T \) where

\[ T = \{ z \in \mathbb{R}^n : \forall i \in \{0, 1, \ldots, n\} : \]

\[ \text{vol}(C(z, z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)) \leq \text{vol}(S), \] (2.4)

then there exists two Euclidean balls of \( B(p, r), B(p, R) \) in \( \mathbb{R}^n \) such that

\[ B(p, r) \subset S \subset T \subset B(p, R), \] (2.5)

where \( p \in \mathbb{R}^n \) is a center with radius \( r \) and \( R \), and \( p \) is also a barycenter of \( S \). It is obvious from (2.3), (2.4) that \( T \subset T_\epsilon \) for a positive value of \( \epsilon \). Let \( \epsilon \) be small, then \( T_\epsilon \) is slightly bigger than \( T \). There also exists a Euclidean ball \( B(p, R') \) with radius \( R' \) and center \( p \) such that \( T_\epsilon \subset B(p, R') \) and \( R' \) is slightly different from \( R \) for some small \( \epsilon \). Finally, we conclude that

\( B(p, r) \subset S \subset rK \subset T_\epsilon \subset B(p, R) \).

**Remark** The value of \( \epsilon \) should be small enough to get the vertices of \( K \) that spanned as large a simplex of \( K \) as possible.

If we use a similarity transformation (as describe in Subsection 1.3.2), we can identify \( \mathbb{R}^n \) with the hyperplane \( \{ (r_j)_{j=0}^n \in \mathbb{R}^n : \sum_{j=0}^n r_j = 1 \} \) in \( \mathbb{R}^{n+1} \) such that \( z_0, \ldots, z_n \) (which are vertices of \( S \)) are a standard basis of \( \mathbb{R}^{n+1} \). Then \( p \) is \( (1/n + 1, \ldots, 1/n + 1) \), and

\[ \frac{R}{r} = \begin{cases} 
\frac{n^2 + 2n}{r}, & \text{if } n \text{ is even;} \\
\frac{n^2 + n^2 - n}{r}, & \text{if } n \text{ is odd.}
\end{cases} \]
Step 4 Examine the Image of \( Z^n \)

Fix \( r \); we want to find an integral vector \( x \) with \( r(x) \in rK \). We are therefore interested in the lattice \( \mathcal{L} = r(Z^n) \). Let \( b_i = r(e_i) \) with \( e_i \) standard basis vectors. Then \( \mathcal{L} \) is a lattice in \( \mathbb{R}^n \) and \( b_1, \ldots, b_n \) is a basis for \( \mathcal{L} \). Let \( d(\mathcal{L}) \) be the determinant of \( \mathcal{L} \) and from the inequality of Hadamard (in Chapter 1)

\[
d(\mathcal{L}) \leq \prod_{i=1}^{n} |b_i|.
\]

We can change the basis of \( \mathcal{L} \) by the reduction process (the details for this process are described in Chapter 3) into one such that the opposite inequality holds:

\[
\prod_{i=1}^{n} |b_i| \leq c_2 d(\mathcal{L}),
\]

where \( c_2 \) is a constant only depending on \( n \), e.g., \( c_2 = c^{n(n-1)/4} \) with \( c \) any constant greater than \( 4/3 \). We shall call a basis \( b_1, \ldots, b_n \) satisfying (2.6) a "Reduced Basis".

Step 5 Concluding Step

We number the reduced basis \( b_1, \ldots, b_n \) such that \( |b_n| = \max\{|b_i| : 1 \leq i \leq n\} \). Then from (1.19) of Lemma 3 in Subsection 1.3.3:

\[
\forall x \in \mathbb{R}^n, \exists y \in \mathcal{L} : |x - y| \leq \frac{1}{2} \sqrt{n} |b_n|.
\]

We will consider the value of \( \frac{1}{2} \sqrt{n} |b_n| \), and divide into 2 cases.

Case 1 If \( \frac{1}{2} \sqrt{n} |b_n| \leq r \), then

\[
\forall x \in \mathbb{R}^n : \exists y \in \mathcal{L} : |x - y| \leq \frac{1}{2} \sqrt{n} |b_n| \leq r.
\]
If we let \( x \) as \( p \) (the barycenter of \( S \)). This means that \( B(p, r) \) is big enough to contain a point of \( \mathcal{L} \), and

\[
y \in B(p, \frac{1}{2} \sqrt{n|b_n|}) \cap \mathcal{L} \subseteq B(p, r) \cap \mathcal{L} \subseteq rK \cap \mathcal{L}.
\]

so \( rK \cap \mathcal{L} \neq \emptyset \). To find \( y \), we can represent \( x \) as a linear combination of the reduced basis such that

\[
x = x_1 b_1 + \cdots + x_n b_n, \quad \text{where } x_i \in \mathbb{R} \text{, and } y = y_1 b_1 + \cdots + y_n b_n, \quad \text{where } y_i \in \mathbb{Z} \text{.}
\]

The problem is to find integral coordinates of \( y \), such that \( |x - y| \leq r \). Let \( z \) be the point at the boundary of \( B(p, r) \), and \( z = z_1 b_1 + \cdots + z_n b_n \), where \( z_i \in \mathbb{R} \). Hence, \( \sum_{i=1}^{n} |(x_i - z_i)b_i|^2 = r^2 \); then to find \( y_i \in \mathbb{Z} \) such that \( \sum_{i=1}^{n} |(y_i - x_i)b_i|^2 \leq r^2 \) is enough to find \( y_i \in \mathbb{Z} \) such that \( y_i \leq z_i \). For example, let \( y_i \) be the nearest integer for \( x_i \), each \( y_i \) is distance from \( x_i \) at most \(|\frac{1}{2}|\), then

\[
\sum_{i=1}^{n} |(y_i - x_i)b_i|^2 \leq \frac{1}{4} \sum_{i=1}^{n} |b_i|^2 \leq \frac{1}{4} \sqrt{n|b_n|} \text{, which is less than or equal to } r. \text{ Hence, } y \in \mathcal{L} \text{ and}
\]

\[
y \in B(p, \frac{1}{2} \sqrt{n|b_n|}) \cap \mathcal{L} \subseteq B(p, r) \cap \mathcal{L} \subseteq rK \cap \mathcal{L}.
\]

Since, we have found a point \( y \) in \( rK \cap \mathcal{L} \), we have found an integer point \( r^{-1}(y) \) in \( K \); we are done.

Case 2. If \( \frac{1}{2} \sqrt{n|b_n|} > r \), this implies that \( B(p, r) \) is too small to be sure of containing a point of \( \mathcal{L} \).

If we write \( \mathcal{L} \) as

\[
\mathcal{L} = \mathcal{L}' + Zb_n \subseteq \mathbb{N} + Zb_n = \bigcup_{k \in \mathbb{Z}} (\mathbb{N} + kb_n).
\]

where \( \mathcal{L}' = \sum_{i=1}^{n-1} Zb_i \), this is a lattice in the hyperplane \( \mathbb{N} = \sum_{i=1}^{n-1} \mathbb{Z}b_i \). Hence, \( \mathcal{L} \) is contained in the union of countably many parallel hyperplanes which have successive distances \( h \) from each other.
where we define $h$ as the distance of $b_a$ to $v$. We are only interested in those hyperplanes that have a non-empty intersection with $rK$, and since $B(p, r) \subset rK \subset B(p, R)$, this implies that these are some of the hyperplanes which intersect $B(p, R)$. Suppose $t$ is the upper bound of the number of the hyperplanes $v + kb_a$ which intersect with $B(p, R)$. Then we have clearly

$$t - 1 \leq \frac{2R}{h}.$$ 

and

$$d(L) = h \cdot d(L')$$

(2.7)

From (2.6), (2.7) and The inequality of Hadamard, applied to $L'$, we get

$$\Pi_{i=1}^{n} |b_i| \leq c_3 \cdot d(L) = c_3 \cdot h \cdot d(L') \leq c_3 \cdot h \cdot \Pi_{i=1}^{n} |b_i|.$$ 

and from $h \leq |b_a|$, (1.20), we have

$$c_3^{-1} |b_a| \leq h \leq |b_a|.$$ 

(2.8)

By $R/r \leq c_1 \cdot \frac{1}{\sqrt{n}} |b_a| > r$, and (2.8), we get

$$2R \leq 2c_1 < c_1 \sqrt{n} |b_a|.$$ 

so

$$t - 1 \leq \frac{2R}{h} < c_1 \sqrt{n} |b_a| < c_1 c_3 \sqrt{n}.$$ 

Hence, the number of values for $k$ that have to be considered is bounded by a constant only depending on $n$. Which values of $k$ need to be considered can easily be deduced from a representation of $p \mid p$ is
the barycenter of $S$ as a linear combination of $b_1, \ldots, b_n$ (the reduced basis). Suppose we found the value of $k$ in the interval $[k_1, k_2]$, then $k$ runs through every integer of this interval. For example, if $k$ is in the interval of $[-2,4]$ then $k$ can be $-2, -1, 0, 1, 2, 3, 4$ and there are 7 hyperplanes of $n$ intersecting $rK$.

**Note.** It will be clear from the example in Section 2.2, how to obtain the values of $k$.

\((**))\) For each fixed value of $k$, we restrict attention to those $y = \sum_{i=1}^{n} y_i b_i$ where $y = (y_1, \ldots, y_n)$ and $y \in \mathcal{L}$. We can replace $x = (x_1, \ldots, x_n)$ of the original constraint problem by $y$, where $y = (y_1, \ldots, y_n) = \sum_{i=1}^{n-1} y'_i k_i + k b_n$. This leads to an integer constraint problem with $(n-1)$ variables $y'_1, \ldots, y'_{n-1}$. We can apply this algorithm for the new problem. Each of the lower dimensional constraint problems treated recursively. The case of dimension 1 or 0 may serve as a basis for the recursion. We can produce an vector $y$ with $y \in B(p,R) \cap \mathcal{L}$. From $rK \subset B(p,R)$ and solve $y$ subject to the constraint problem, hence

$$y \in rK \cap \mathcal{L} \subset B(p,R) \cap \mathcal{L}$$

Since we have found a point $y$ in $rK \cap \mathcal{L}$, we have found an integer point $r^{-1}(y)$ in $K$; we are done. The problem of solving for $y$ in the constraint problem is an equivalent problem to the original constraint problem with a property that the last coordinate of $y$ belongs to an interval whose length is bounded by the value of $k$, where $k \leq t - 1 \leq c_1 c_2 \sqrt{n}$, so $k$ is a constant only depending on $n$.

If there doesn't exist such $y$ for that particular value of $k$, such that $y \in rK \cap \mathcal{L}$, then select a new fixed integer value of $k$ in the interval $[k_1, k_2]$. Repeat (**)) for each value of $k$, until discover that there exists or there doesn't exist such $y \in rK \cap \mathcal{L}$. If there exists at least one vector of $y$, then terminate. If there doesn't exist such $y$ for every possible value of $k$, then we conclude that our constraint problem...
has no integer solution.

Note. It might happen that, for some fixed $k$ the new problem is infeasible. Terminate from that $k$ and select a new value of $k$. 
2.2 An Example of the algorithm in Section 2.1

The following is an example of the algorithm in Section 2.1. The purpose of this algorithm is to find whether there is an integer point in the feasible region, that is to find \( x \) as a integer vector such that \( Ax \leq b \). As mentioned in the Introduction, this is a feasibility problem. Step 1-Step 4 are preparation steps. The results from these steps are used to transform the problem into a new problem.

To solve the new problem is to find a vector \( y \) such that \( A'y \leq b' \), where

\[
A' = \begin{bmatrix} A \\ I \end{bmatrix}, \quad b' = \begin{bmatrix} b \\ 0 \end{bmatrix}
\]

and \( y \) is a point in a lattice. Step 5 is a concluding step, either there exists a vector \( y \) in a lattice, or we need to construct another problem with one dimension less than the original problem, and return to Step 1 with the new problem. Hence we can decide recursively the solution to the original problem.

In the example, we will illustrate step by step the algorithm of Section 2.1. Even if in Step 1 we find an integer solution satisfy the constraints problem, we will continue through the calculation, so that it is clear how each step works. In this example, a complete calculation is given for a \( d \)-dimension problem only, where \( d \) is less than or equal to the number of variables. The problems of the lower dimension can be solved in the same way.

**PROBLEM**

Inequality constraints are as follows:

\[
x_1 + 2x_2 + x_3 \leq 4
\]

\[
x_1 + 2x_2 + 2x_3 \leq 6
\]

\[
x_1, x_2, x_3 \geq 0.
\]
Figure 2-1: The feasible region of this problem

**Step 1** Find a Simplex in $K$

Use the Simplex method to solve LP problems.

1.1) Maximize any arbitrary linear function, let us say

$$\max x_1$$

subject to

$$x_1 + 2x_2 + x_3 \leq 4$$

$$x_1 + 2x_2 + 2x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0.$$

An optimal solution is $x_1 = 4, x_2 = 0, x_3 = 0$, then $v_0 = (4, 0, 0)$.

1.2) Maximize another arbitrary linear function,

$$\max x_3$$
subject to \( x_1 + 2x_2 + x_3 \leq 4 \)

\[
x_1 + 2x_2 + 2x_3 \leq 6
\]

\[x_1, x_2, x_3 \geq 0.\]

An optimal solution is \( x_1 = 0, x_2 = 0, x_3 = 3 \); so that \( v_1 = (0, 0, 3) \).

1.3) Find the equation of a plane passing through two vertices \( v_0 = (4, 0, 0) \) and \( v_1 = (0, 0, 3) \).

The equation is \( 3x_1 + 4x_2 - 12 = 0 \), then our problem is

\[
\text{max} \quad 3x_1 + 4x_2
\]

subject to \( x_1 + 2x_2 + x_3 \leq 4 \)

\[
x_1 + 2x_2 + 2x_3 \leq 6
\]

\[x_1, x_2, x_3 \geq 0.\]

An optimal solution is \( x_1 = 2, x_2 = 0, x_3 = 2 \); then \( v_2 = (2, 0, 2) \), and \((v_2 - v_0), (v_1 - v_0)\) are linearly independent.

1.4) The equation of the plane spanned by 3 points \( v_0, v_1, v_2 \) is \(-2x_2\). Then we solve the problem of

\[
\text{min} \quad -2x_2 \quad \text{or max} \quad 2x_2
\]

subject to \( x_1 + 2x_2 + x_3 \leq 4 \)

\[
x_1 + 2x_2 + 2x_3 \leq 6
\]

\[x_1, x_2, x_3 \geq 0.\]
An optimal solution is \( x_1 = 0, x_2 = 2, x_3 = 0 \); then \( v_3 = (0, 2, 0) \) and \( (v_1 - v_0), (v_2 - v_0) \) are linearly independent.

1.5) In the original problem, the number of variables \( n \) is 3; from above, there are 4 vertices such that \( (v_3 - v_0), (v_2 - v_0), (v_1 - v_0) \) are linearly independent and the dimension is 3. We conclude that the convex hull \( K \) contains \( v_0, v_1, v_2, v_3 \) and \( K \) has a positive volume.

Step 2 Find a large Simplex inside \( K \)

The convex hull (\( K \)) contains

\[
\begin{align*}
v_0 &= (4, 0, 0) \\
v_1 &= (0, 0, 3) \\
v_2 &= (2, 0, 2) \\
v_3 &= (0, 2, 0).
\end{align*}
\]

These vertices form a 3-dimensional simplex in \( K \); let us call it \( D \): volume of \( D \) or \( \text{vol} \ D \) is \( 2/3 \).

Remark:

\[
\text{volume of a-dimensional simplex } (v_0, \ldots, v_n) = \frac{1}{n!} \left| \det \left( \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ v_0 & v_1 & \cdots & v_n \end{array} \right) \right|.
\]

2.1) For \( i = 0 \), let \( v_0 = (x_1, x_2, x_3) \); then the volume of a new \( D \) composed of \( v_0 = (x_1, x_2, x_3) \), \( v_1 = (0, 0, 3), v_2 = (2, 0, 2), v_3 = (0, 2, 0) \) is \( 1/6 \left( -2x_1, -6x_2, 4x_3 + 12 \right) \). We will solve two LP problems: maximizing and minimizing the volume of the new \( D \) on \( K \). Check for each new \( v_0 \), compare the new volume \( D \) with the old one. Replace \( v_0 \) with \( x_0, x_1, x_3 \), if the volume of a new \( D \) is greater than or equal to \( (1 + \epsilon) \text{vol} \ D \), where \( \epsilon \) is some positive real number, otherwise \( v_0 \) is unchanged. Let \( \epsilon = 0.1 \).
2.1.1) Solve the problem of

\[
\max \quad \frac{1}{3} x_1 - x_2 - \frac{2}{3} x_3
\]

subject to

\[
x_1 + 2x_2 + x_3 \leq 4
\]

\[
x_1 + 2x_2 + 2x_3 \leq 6
\]

\[
x_1, x_2, x_3 \geq 0.
\]

An optimal solution is \((0,0,0)\).

2.1.2) Solve the problem of

\[
\min \quad \frac{1}{3} x_1 - x_2 - \frac{2}{3} x_3
\]

subject to

\[
x_1 + 2x_2 + x_3 \leq 4
\]

\[
x_1 + 2x_2 + 2x_3 \leq 6
\]

\[
x_1, x_2, x_3 \geq 0.
\]

An optimal solution is \((0,1,2)\).

2.1.3) Check the volume of the new \(D\).

a) Volume of \(D\) spanned by \(v_0 = (0,0,0), v_1 = (0,0,3), v_2 = (2,0,2), v_3 = (0,2,0)\) is 2.

b) Volume of \(D\) spanned by \(v_0 = (0,1,2), v_1 = (0,0,3), v_2 = (2,0,2), v_3 = (0,2,0)\) is \(1/3\).

2.1.4) Volume of \(D\) with \(v_0 = (0,0,0) \geq (1 + 0.1)\) volume of \(D\) with \(v_0 = (4,0,0)\):

and volume of \(D\) with \(v_0 = (0,1,2) < (1 + 0.1)\) volume of \(D\) with \(v_0 = (4,0,0)\).

Thus, we now replace \(v_0\) with \((0,0,0)\); then the 3-simplex \(D\) spanned by

\[v_0 = (0,0,0)\]
\( v_1 = (0, 0, 3) \)

\( v_2 = (2, 0, 2) \)

\( v_3 = (0, 2, 0) \).

2.2) Repeat Step 2.1 for \( i = 1, 2, 3 \). Finally, a 3-dimensional simplex \( D \) is spanned by

\( v_0 = (0, 0, 0) \)

\( v_1 = (4, 0, 0) \)

\( v_2 = (0, 0, 3) \)

\( v_3 = (0, 2, 0) \).

and the volume of \( D \) is 4.

step 3 Mapping the Simplex to a regular Simplex

3.1) Select a regular 3-dimensional simplex (\( S \)) with vertices

\( z_0 = (0, 0, 0) \)

\( z_1 = (12, 12, 0) \)

\( z_2 = (0, 12, 12) \)

\( z_3 = (12, 0, 12) \).

and the length of this regular simplex (\( S \)) is 16.97.

3.2) Find a mapping \( r \) such that \( (z_i - z_0) = r(v_i - v_0) \) for \( 1 \leq i \leq 3 \). Then

\[
    r = \begin{pmatrix} 3 & 0 & 4 \\ 3 & 6 & 0 \\ 0 & 6 & 4 \end{pmatrix}
\]
3.3) There exist two Euclidean balls such that

\[ B(p, r) \subset S \subset T \subset B(p, R). \]

where \( r \) is the radius of a ball contained inside \( S \), \( R \) is the radius of a ball surrounding \( T \). We can find the values of \( r \) and \( R \) by transforming \( S \) into a standard regular simplex \( S' \) in \( \mathbb{R}^n \), such that

\[
\begin{align*}
z_0' &= (1, 0, 0, 0) \\
z_1' &= (0, 1, 0, 0) \\
z_2' &= (0, 0, 1, 0) \\
z_3' &= (0, 0, 0, 1)
\end{align*}
\]

are the vertices of \( S' \) and its edge length is 1.414.

By Lemma 2 of Subsection 1.3.3:

\[ B(p, r') \subset S' \subset B(p, R'). \]

with

\[
r' = \sqrt{\frac{1}{n(n+1)}}
\]

\[
R' = \begin{cases} 
\sqrt{\frac{n^2+3n}{n+1}} & \text{if } n \text{ is even} \\
\sqrt{\frac{n^2+n-1}{n+1}} & \text{if } n \text{ is odd}
\end{cases}
\]

Hence, for \( n = 3 \), \( r' = 0.289 \), \( R' = 1.66 \) and

\[
r = r' \times \frac{\text{edge length of simplex } S}{\text{edge length of simplex } S'}
\]

\[
= 0.289 \times \frac{16.97}{1.414}
\]

\[
= 3.46.
\]
\[ R = R' \times \frac{\text{edge length of simplex } S}{\text{edge length of simplex } S'} \]
\[ = 1.66 \times \frac{16.97}{1.414} \]
\[ = 19.92 \]

**Step 4** Examine the image of \( Z^* \)

4.1 Find the basis \( b_i (1 \leq i \leq 3) \) for a lattice \( \mathcal{L} \) such that \( \mathcal{L} = rZ^* \). Let \( e_i, 1 \leq i \leq 3 \) be the standard basis for \( Z^* \). Then \( b_i = r(e_i) \) and

\[ b_1 = (3, 3, 0) \]
\[ b_2 = (0, 6, 6) \]
\[ b_3 = (4, 0, 4) \]

4.2 Find the reduced basis for a lattice. We can find a reduced basis by using an computer (the implementation for both algorithms of the reduction process are obtained in Chapter 3). Thus, the reduced basis for this problem is

\[ b_1 = (3, 3, 0) \]
\[ b_2 = (-4, 6, 2) \]
\[ b_3 = (1, -3, 4) \]

**Step 5** Concluding Step

Renumber the reduced basis such that \(|b_3| = \max\{|b_i| : 1 \leq i \leq 3\}\). Thus, the reduced basis is

\[ b_1 = (3, 3, 0) \text{ norm } = 4.24, \]
\[ b_2 = (1, -3, 4) \text{ norm } = 5.66. \]
\[ b_2 = (-4, 6, 2) \text{ norm } = 7.28. \]

We transform the problem \( Ax \leq b \) into \( A'y \leq b \), where \( x \in \mathbb{R}^n, (y = rx) \) and \( y \in \mathcal{L} \). Then

\[ A' = Ar^{-1}. \]

In this example:

\[
A = \begin{bmatrix}
1 & 2 & 1 \\
1 & 2 & 2 \\
\end{bmatrix}, \quad r^{-1} = \begin{bmatrix}
1/6 & 1/6 & -1/6 \\
-1/12 & 1/12 & 1/12 \\
1/8 & -1/8 & 1/8 \\
\end{bmatrix}.
\]

Then

\[
A' = \begin{bmatrix}
1/8 & 5/24 & 1/8 \\
1/4 & 1/12 & 1/4 \\
\end{bmatrix}.
\]

Problem 1 (original problem) \hspace{1cm} Problem 2 (transformed problem)

\[
x_1 + 2x_2 + 2x_3 \leq 4 \hspace{1cm} 1/8y_1 + 5/24y_2 + 1/8y_3 \leq 4
\]

\[
x_1 + 2x_2 + 2x_3 \leq 6 \hspace{1cm} 1/4y_1 + 1/12y_2 + 1/4y_3 \leq 6
\]

\[ x_1, x_2, x_3 \geq 0, \text{ integers} \hspace{1cm} y_1, y_2, y_3 \geq 0, \in \mathcal{L} \]

Consider the value of \( \frac{1}{3}(\sqrt{n}||b||) \)

For \( n = 3, ||b|| = 7.48 \), then \( \frac{1}{3}\sqrt{3}(7.48) = 6.48 \). From Step 5, \( r = 3.46 \). Hence, \( \frac{1}{3}\sqrt{3}(7.48) > r \).

Then we are in the situation of Case 5.2.

a) Let \( p \) be a barycenter of \( S \), then

\[ p = \frac{1}{4} \{(0, 0, 0) + (12, 12, 0) + (12, 0, 12) + (0, 12, 12)\} = (6, 6, 6). \]

Represent \( p \) as a linear combination of the reduced basis, which is

\[
\frac{9}{4}b_1 + \frac{5}{4}b_2 + \frac{1}{2}b_2.
\]
b) Find the actual number of hyperplanes (\( \mathcal{H} = \sum_{i=1}^{n-1} \mathbb{R}b_i \)) intersecting \( B(p,R) \) with radius \( R = 19.92 \).

From Figure 2.2, if \( M \) is a point at the boundary of \( B \), then we can represent \( M \) as a linear combination of the reduced basis vectors such that

\[
M = (m_1, m_2, m_3) = \frac{9}{4} b_1 + \frac{5}{4} b_2 + k' b_3
\]

where \( k' \) is any real number. Then

\[
(m_1 - 6)^2 + (m_3 - 6)^2 + (m_3 - 6)^2 \leq (R)^2 = (19.92)^2
\]

\[
(8 - 4k' - 6)^2 + (3 + 6k' - 6)^2 + (5 + 2k' - 6)^2 \leq 396.81
\]

\[-2.16 \leq k' \leq 3.10.
\]

Since we represent \( \mathcal{L} \) such that \( \mathcal{L} = \bigcup_{k \in \mathcal{L}} (\mathcal{X} + k b_n) \), there are 6 hyperplanes of \( \mathcal{L} \) intersecting to \( B(p,R) \) and \( k = \{3, 2, 1, 0, -1, -2\} \).

c) Represent \( y \) as a linear combination of the reduced basis vectors,

\[
y = (y_1, y_2, y_3) = y_1 b_1 + y_2 b_2 + y_3 b_3.
\]
Fix the value of $y_3$, select from one of $k$. Hence, $y = y_1 b_1 + y_2 b_2 + 3 b_3$, and

$$y_1, y_2, y_3 = (3y_1 + y_2 - 12), (3y_1 - 3y_2 + 18), (4y_2 + 6)$$  \hspace{1cm} (2.9)$$

Substituting $y$ from (2.9) into Problem 2, we now obtain Problem 3 such that

**Problem 3**

$$y_1' \leq 1$$

$$y_1' + y_2' \leq 6$$

$$3y_1' + y_2' \geq 12$$

$$3y_1' - 3y_2' \geq -18$$

$$y_2' \geq -\frac{3}{2}$$

$$y_1', y_2' \in \mathbb{Z}.$$  

We now return to Step 1 with the new integer programming problem with only two variables. Apply the algorithm to this new problem. We treat the lower dimension programs recursively, then we can decide the problem. If this problem has no solution (no such $y \in \mathbb{Z}$), then we fix a new value of $k$, obtain a new problem with two variables and repeat the process.

Finally, we can decide that the transformation problem has a solution or has no solution. If the transformation problem has a solution (at least one), this implies that the original integer linear program has at least one integer point in its feasible region, and we can obtain $x$ which is a solution for the original problem from $x = r^{-1} y$. If the transformed problem has no solution then the original problem has no integer point (not even one) in its feasible region. From the next section, Lenstra [12], has shown that the running time of this algorithm is bounded by a polynomial in the input length with a fixed value of number of variables. This finishes the example of the algorithm.
2.3 Analysis of the algorithm

We measure the complexity (the amount of work done) of an algorithm as a function of the size of the input. The input in combinatorial optimization problems is a combinatorial object - a graph, a set of integers, a family of finite sets, and so on. We must somehow encode it or represent it as a sequence of symbols over some fixed alphabet such as bits or ASCII characters. We define the size of the input to be the length of this sequence, that is, the number of symbols in it. We define the complexity of the algorithm for the input size to be the worst-case behavior of the algorithm.

If \( f(n) \) and \( g(n) \) are functions from the positive integers to the positive reals, we define \( f(n) = O(g(n)) \) if there exists a constant \( c > 0 \) such that for large enough \( n \), \( f(n) \leq cg(n) \). We speak of \( f(n) = O(g(n)) \) as the growth rate of a function \( f(n) \) and \( O \) as "big-oh".

An algorithm is a practically useful solution to a computational problem only if its complexity grows polynomially with respect to the size of the input. The complexity may not be a polynomial itself but if bounded by a polynomial, this also qualifies. Examples of such polynomially bounded functions are \( n^3 \) and \( n \log n \).

Some general rules for analysis of the algorithm

1) The running time of each assignment, read and write statement can usually be taken to be \( O(1) \).

2) The running time of a sequence of statements is determined by the sum rule. That is, the running time of the sequence is the largest running time of any statement in the sequence. For example, suppose we have three steps whose running times are respectively, \( O(n^2) \), \( O(n^3) \), \( O(n \log n) \). Then the running time of the first two steps executed sequentially is \( O(\max(n^2, n^3)) \) which is \( O(n^3) \). The running time of all three together is \( O(n^3, n \log n) \) which is \( O(n^3) \).
3) The running time of an if-statement is the cost of the conditionally executed statements plus the time for evaluating the conditions. The time to evaluate the condition is normally $O(1)$. The time for an if-then-else construct is the time to evaluate the condition plus the larger of the time needed for the statements executed when the condition is true and the time for the statements executed when the condition is false.

4) The time to execute a loop is the sum over all times around the loop of the time to execute the body and the time to execute the condition for termination (usually the latter is $O(1)$). Often this time is up to a constant factor, the product of the number of times around the loop and the largest possible time for the one execution of the body, but we must consider each loop separately to make sure. The number of iterations around a loop is usually clear, but there are times when the number of iterations can not be computed precisely. It could even be that the program is not a finite algorithm, and there is no limit to the number of times we go around certain loops.

The rule for products is the following: if $T_1(n)$ and $T_2(n)$ are $O(f(n))$ and $O(g(n))$, respectively, then $T_1(n)T_2(n)$ is $O(f(n)g(n))$, and it follows from the product rule that $O(cf(n))$ means the same things as $O(f(n))$, if $c$ is any positive constant.

5) If we have a program with procedures, none of which is recursive, then we can compute the running time of the various procedures one at a time, and use the rule of sum and the rule of product to evaluate the running time.

6) If there are recursive procedures then we can not find an ordering of all procedures so that each calls only previously evaluated procedures. What we must do is associate with each recursive procedure an unknown time function $T(n)$, where $n$ measures the size of the arguments to the procedure. We...
can get a recurrence for $T(n)$, that is, an equation of $T(n)$ in terms of $T(k)$ for various values of $k$.

For example, we might get

$$T(n) = \begin{cases} c + T(n-1), & \text{if } n > 1 \\ d, & \text{if } n \leq 1. \end{cases}$$

where $c, d$ are some constants.

**Complexity of the algorithm**

We first define the length of the input for a constraint problem with a fixed $n$. Since $A$ is an $(m \times n)$-matrix and $b$ is an $m$-vector, there are $(n+1)m$ coordinates. Let $a$ denote the maximum of the absolute values of the coefficients of $A$ and $b$. Then each coordinate has at most a constant times $\log(a+2)$ binary digits, so we can take the length of the input $l$ as $(n+1)m\log(a+2)$ or $nm\log(a+2)$.

From Step 1 and Step 2, we solve several linear programming problems to get $v_0, v_1, \ldots, v_M$. By [10], Khachian shows that each linear programming can be solved within a polynomial time in $l$. In fact we need $O(n^2(n^2 + m)l)$. Hence, in these 2 steps, the running time is a polynomial bounded by a function of $l$. From Step 3, Lovasz, [7] has shown that a transformation $r$ can be found in polynomial time bounded by a function of $l$, even for very large $n$.

In Step 4, as will be seen later in Chapter 3, there are two algorithms for finding a reduced basis. Even though the complexity of the reduction process is bounded by polynomial time of its input length, but it is difficult to compute its complexity precisely. Therefore we will not include the complexity for the reduction process in computing the complexity of this algorithm. We consider the reduction process II instead, which is $O(n^4 \log B)$ where $B$ is the upper bound of the norm of input basis. Since the input basis is computed from the convex set $K$, which are the optimum solutions of several LP problems.
each optimal solution is bounded by the length of input (l), [3], hence \( B \) is also bounded by the input length (l). Thus, the complexity of Step 4 is bounded by a polynomial time in the length of input (l).

For Step 5, if the algorithm terminates in Case 1, then it is clear that this step can be done in a polynomial time in \( l \). In Case 2, we construct a new constraint problem with only \((n - 1)\) variables, and its input size is bounded by the input size of the previous one. Therefore, in Case 2, its complexity depends on Step 1 to Step 4, which is a polynomial time bounded by \( l \). Since each step is a polynomial time bounded by \( l \), thus, in Case 2, we can finish in a time bounded by a polynomial in \( l \).

From the above information, each step can be done within a polynomial time of \( l \). Therefore, from the rule of summation, the complexity of this algorithm is bounded by a polynomial function of its input length.

It is very difficult to compute the actual complexity, especially in the worst case performance. From Step 5, it is not clear how many new integer programming problems we need to construct in order to get an integer point. Its complexity might be a very large polynomial of function. In fact, this algorithm is not practical for a large value of \( n \); the cutting plane method or branch and bound method also work well. Consider Steps 1, 2, 3; we computed their complexity based on an ellipsoid method which is not a practical method also. In general, this algorithm is useful only from a theoretical point of view.
Chapter 3

A REDUCTION PROCESS FOR LATTICES

In Chapter 2, we referred to a process that finds a reduced basis for a lattice. This process changes a given basis for a lattice \( \{ \mathcal{L} \} \) in \( \mathbb{R} \) into a reduced basis \( \{ b_i \} \) of \( \mathcal{L} \). We shall define any basis as a reduced basis if

\[
\prod_{i=1}^{n} |b_i| \leq c_2 d(\mathcal{L}).
\]

(3.1)

where \( \{ b_i \} \) is a basis for \( \mathcal{L} \), \( 1 \leq i \leq n \), \( c_2 \) is a constant only depending on \( n \), \( d(\mathcal{L}) \) is a determinant of \( \mathcal{L} \).

We will describe two algorithms for changing a given basis for \( \mathcal{L} \) to a reduced one. The algorithms are given in Section 3.1 and Section 3.2. The first algorithm, we shall call the Reduction process I, is a recursive algorithm. This is an algorithm by Lenstra - see [12], its complexity is bounded by a polynomial function of the input length only for a fixed value of \( n \). Whereas the second one (the reduction process II) is a non-recursive one, this is an algorithm by Looijen - see [13], its complexity is bounded by a polynomial function of its input length even for a vary \( n \). The second algorithm is an improvement of the first one.

At the end of this Chapter, we include the implementation of both algorithms, some results and also some limitations.
3.1 The reduction process I

Let \( b_1, \ldots, b_n \in \mathbb{R}^n \) be \( n \) linearly independent vectors. Put \( \mathcal{L} = \sum_{i=1}^n Zb_i \); this is a lattice in the linear subspace \( \sum_{i=1}^n \mathbb{R}b_i \) and \( d(\mathcal{L}) = |\text{det}(b_1, \ldots, b_n)| \). If we regard the \( b_1, \ldots, b_n \) in \( \text{det}(b_1, \ldots, b_n) \)
first as rows and then as columns and multiply the two determinants together, we have

\[
|\text{det}(b_1, \ldots, b_n)|^2 = \text{det}\{\langle b_i, b_j \rangle\}_{1 \leq i, j \leq n}.
\]

The reduction process is as follows

\textbf{Step 1}

If \( n > 1 \) then renumber the \( b_i \) such that \( |b_1| = \min\{|b_i| \mid 1 \leq i \leq n\} \), and let \( \mathcal{V} \) be the hyperplane in \( \mathbb{R}^n \) orthogonal to \( b_1 \) so that \( \mathcal{V} = \{x \in \mathbb{R}^n : \langle x, b_1 \rangle = 0\} \). We can take the projection \( \overline{b}_1 \) of \( b_1 \) on \( \mathcal{V} \) namely

\[
\overline{b}_1 = b_1 - \frac{\langle b_1, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 \quad \text{for} \ (2 \leq i \leq n),
\]

and put \( \mathcal{L} = \sum_{i=2}^n Z\overline{b}_i \). Then \( \mathcal{L} \) is a lattice in the \((n-1)\)-dimensional vector space \( \mathcal{V} = \sum_{i=1}^n \mathbb{R}\overline{b}_i \)
and \( d(\mathcal{L}) = \frac{d(\mathcal{L})}{|b_1|} \). Repeat this step recursively until \( n = 1 \); for \( n = 1 \), we have \( d(\mathcal{L}) = |b_1| \)

\textbf{Step 2}

We can change the basis \( \overline{b}_2, \ldots, \overline{b}_n \) for \( \mathcal{L} \) into a basis \( \overline{b}'_2, \ldots, \overline{b}'_n \) for \( \mathcal{L} \) satisfying (3.1). The new basis can be written in terms of the old one by \( \overline{b}_i = \sum_{j=1}^n m_{ij} b_j \), with \( m_{ii} \in \mathbb{Z} \), and its determinant \( \text{det}(m_{ij})_{2 \leq i, j \leq n} = \pm 1 \). Now we can write \( b_1' = \sum_{j=1}^n m_{ij} b_j \) (for \( 2 \leq i, j \leq n \)). Then \( b_1, b_2', \ldots, b_n' \) is a basis for \( \mathcal{L} \) and \( b_1' \) is the projection of \( b'_1 \) on \( \mathcal{V} \):

\[
\overline{b}'_1 = b'_1 - \frac{\langle b_1', b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 \quad (2 \leq i \leq n).
\]
Let \( n \) be the integer nearest to \( \frac{(b'_1 b_1)}{(b_1 b_1)} \), and put \( b''_i = b'_i - n b_1 \), \( 2 \leq i \leq n \). Then \( b_1, b''_2, \ldots, b''_n \) is a basis for \( \mathcal{L} \).

We show how to get a matrix \( M \) with \( m_{ij} \in \mathbb{Z} \) and its determinant is \( \pm 1 \). In the case of \( n = 1 \), we start \( M \) with the identity matrix; for \( n > 1 \), we can generate a new matrix \( M \) from the previous \( M \) at the \( 1 \) dimension lower and \( n \), such that

\[
\text{a matrix } M \text{ at } k\text{-dimension } \mathcal{L} = \begin{bmatrix} 1 & 0 \\ n & \text{a matrix } M \text{ at } k-1 \text{ dimension } \mathcal{L} \end{bmatrix}
\]

for \( 2 \leq i \leq k 

\textbf{Step 3}

At each \( k \)-dimensional \( \mathcal{L} \), we get \( b_1, b''_2, \ldots, b''_n \) from \textbf{Step 2}. We need to check whether \( b_1, b''_2, \ldots, b''_n \) for \( 1 \leq k \leq n \) is a reduced basis (i.e., satisfies the equation (3.1)). Let \( c' = \frac{1}{2} (c - 1)^{-1} \), where \( c > \frac{3}{4} \), therefore \( 0 < c' < \frac{3}{4} \). We distinguish two cases.

\textbf{Case 1} If \( c' |b_i|^2 \leq |b_1|^2 \) for \( i = 2, \ldots, k \). Hence, the basis \( b_1, b''_2, \ldots, b''_n \) for \( \mathcal{L} \) satisfies (3.1), and \( b_1, b''_2, \ldots, b''_n \) is a reduced basis for the \( k \)-dimensional lattice \( \mathcal{L} \). If \( k = n \) the algorithm terminates.

\textbf{Case 2} If \( c' |b_i|^2 > |b_1|^2 \) for \( 2 \leq i \leq k \). Hence, the shortest vector in the basis \( b_1, b''_2, \ldots, b''_n \) for the \( k \)-dimensional \( \mathcal{L} \) is substantially shorter than the shortest vector \( b_1 \) in the basis \( b_1, \ldots, b_k \). Now we return to the beginning of the algorithm \textbf{Step 1} with \( b_1, \ldots, b_k \) replaced by \( b_1, b''_2, \ldots, b''_n \).

This finishes the algorithm. The following is the proof of how the algorithm can terminate in \textbf{Case 1}.

\textbf{Proof that the Basis in Case 1 is Reduced}

To prove that \( b_1, b''_2, \ldots, b''_n \) for \( \mathcal{L} \) satisfies (3.1). Since \( \mathcal{L} \) is a lattice in the \((n - 1)\)-dimensional vector space \( \sum_{i=1}^{n} \mathbb{R}b_i \), and

\[
d(\mathcal{L}) = \frac{d(\mathcal{L})}{|b_1|}.
\]

(3.2)
From Step 2

\[ b''_i = b'_i - n_i b_i \]

where \( n_i \) = integer nearest to \( \frac{b'_i}{b_i} \).

\[ \text{and} \quad b'_i = b'_i + \frac{(b'_i, b_1)}{(b_1, b_1)} b_1 \]

From the above equations, we get

\[ b''_i = b'_i + r_i b_1 \quad \text{for some} \quad r_i \in \mathbb{R}, \quad |r_i| \leq 1/2, \quad \text{and} \quad 2 \leq i \leq n \]

Since, \((b'_i, b_1) = 0\), this implies that

\[ |b''_i|^2 \leq |b'_i|^2 + \frac{1}{4} |b_1|^2 \quad (3.3) \]

By the condition in Case 1, \(c' |b_1|^2 \leq |b_i|^2\) for \(i = 2, \ldots, n\) and (3.3), we get

\[ |b''_i|^2 \leq |b'_i|^2 (1 + \frac{1}{4} c'^{-1}) = c |b'_i|^2 \quad (3.4) \]

since \( c' = \frac{1}{4} (c - 1)^{-1} \) or \( c = (1 + \frac{1}{4} c'^{-1}) \). We want to prove that:

\[ \Pi_{i=1}^n |b_i| \leq c_2 d(\mathcal{L}) \leq c^n (n-1)^{1/4} d(\mathcal{L}), \quad \text{where} \quad c > \frac{4}{3} \]

We prove this by induction. For \( n = 1 \) it is obvious. Suppose it true for \( \mathcal{L} \) (\( n - 1 \) dimension).

From Step 2, we changed the basis \( b_2, \ldots, b_n \) for \( \mathcal{L} \) into a basis \( b'_2, \ldots, b'_n \) for \( \mathcal{L} \) satisfying

\[ \Pi_{i=1}^n |b'_i| \leq c'_2 d(\mathcal{L}) \quad (3.5) \]

where \( c'_2 = c^{(n-1)(n-2)} \) from (3.4)

\[ \Pi_{i=1}^n |b''_i|^2 \leq c^{n-1} \Pi_{i=1}^n |b'_i|^2 \]
\[
\Pi_{i=1}^{n} |b_i''| \leq c^{-1} \cdot \Pi_{i=1}^{n} |b_i'|.
\]

Then (3.5) implies that

\[
|b_i' \Pi_{i=1}^{n} |b_i''| \leq c^{-1} \cdot |b_i'| \Pi_{i=1}^{n} |b_i'|
\]

\[
|b_i' \Pi_{i=1}^{n} |b_i''| \leq c^{-1} \cdot c^{-1} \cdot \cdots \cdot d(\mathcal{L})
\]

\[
\leq c^{-n+1} \cdot d(\mathcal{L})
\]

Hence the basis \( b_1, b_2', \ldots, b_n'' \) for \( \mathcal{L} \) satisfies the required inequality.

For case 2 of Step 5, since \( c', |b_i'| \geq |b_i''| \) for some \( i \in \{2, \ldots, n\} \) Then for this \( i \) we have by

(3.3)

\[
|b_i''| \leq (c' + \frac{1}{4}) |b_i'|.
\]

where \( (c' + \frac{1}{4}) < 1 \) since \( 0 < c' < \frac{1}{4} \). This contradicts the fact that \( b_1 \) is a vector with the minimum norm. Then we return to Step 1 with a new basis. This finishes the proof.

Complexity of the algorithm

Let \( n \) and \( c \) be fixed and the initial basis vectors for a lattice \( \mathcal{L} \) have rational coefficients. We denote these initial vectors by \( b_i^0 \), \( 1 \leq i \leq n \), in order to avoid confusion with the changing meaning of \( b_i \) during the several passes of the algorithm.

The basic operation for this algorithm is the number of times that we loop through the algorithm.

This implies that we only consider the number of times passing Case 2.

Let \( a_1 \) denotes the maximum absolute value of the numerators and denominators of the coefficients of \( b_1' \) and \( b_i' \). and let \( \text{m}(\mathcal{L}) = \min \{|x| \mid x \in \mathcal{L}, x \neq 0\} \). From (3.6), the worst case that can happen is that we always get \( |b_i''| \) is less than but almost equal to \( (c' + 1/4) |b_i'| \). This is a situation of Case 2, so we keep on reducing the basis in that dimension until \( |b_i''| = \text{m}(\mathcal{L}) \). This situation can happen in every
lower dimension. If we assume that it is an $n$-dimensional, then we replace $b_i$ by $b_i'$, and reduce $b_i$. Let $u = c' + 1/4$ and $|b_i'|^2 = K$. Each times $|b_i'|^2$ is reduced by a factor of $u$ (since $|b_i'|^2$ is almost equal to $|b_i|^2$) the sequence of $|b_i'|^2$ generated from the reduction process will be $K, Ku, Ku^2, \ldots, Ku^r$. If $Ku^r = m(L)$, then we stop the process, and $r$ is the number of times passing through Case 2 for the current basis $n$. Hence,

$$r = \frac{\log(|b_i'|^2/m(L))}{\log(u)}$$

Since this situation can occur in every lower dimension, we can conclude that the complexity of this algorithm is a function bounded by $\log(|b_i'|^2/m(L)) / \log(u)$.

Consider the value $m(L)$, we can use the lower bound for $m(L)$ such that $m(L) \geq den(L)^{-1}$, where $den(L)$, the denominator of $L$, denotes the least positive integer $k$ with $L \in \frac{1}{k} \mathbb{Z}^n$. It is the least common multiple of the denominators of the coefficients of $b_i$. Thus,

$$\frac{\log(|b_i'|^2/m(L))}{\log(u)} \leq \frac{\log(|b_i'|^2/den(L)^{-1})}{\log(u)}$$

Let $a_1 = p/q$, where $a_1$ is a maximum of absolute values of the coefficients $b_i$. Then

$$na_1^2 = np^2/q^2. \text{ Clearly, } np^2/q^2 > |b_i| > 0. \text{ Then, }$$

$$\frac{c'' \log(|b_i'|^2/den(L)^{-1})}{\log(u)} \leq \frac{c'' np^2}{q^2} den(L)^{-1}$$

where $c'' = 1/|\log(u)|$. The value of $a_1$ is a factor of $den(L)$ and we can write $den(L)$ as a function of $a_1$. Thus, $c'' np^2/q^2 den(L)^3$ is a polynomial function of $\log a_1$. We conclude that the complexity of this algorithm is bounded by a polynomial in the length of its input.

It is difficult to calculate the actual work for this algorithm, especially in the worst case. The complexity could take a very large polynomial function of the number of constraints if we fixed the number of
variables, but the constant term might be an exponential function of the number of variables. There still is room for improvement. Lovász [13] introduced The Reduction Process II, which has running time bounded by a polynomial function in the length of the input even for the varying number of constraints and variables. We will present in the next section.
3.2 The reduction process II

Let \( \mathcal{L} \) be an \( n \)-dimensional lattice in \( \mathbb{R}^n \), \( n \) a positive integer. Then \( \mathcal{L} \) has the structure

\[
\mathcal{L} = \sum_{i=1}^{n} \mathbb{Z}b_i = \left\{ \sum_{i=1}^{n} m_i b_i : m_i \in \mathbb{Z} \right\} \quad (1 \leq i \leq n),
\]

and \( b_1, \ldots, b_n \) is a basis for \( \mathcal{L} \). In this algorithm, a reduced basis for \( \mathcal{L} \) is defined in such a way that

\[
|\mu_{ij}| \leq \frac{1}{2} \quad (1 \leq j < i \leq n), \tag{3.7}
\]

and

\[
|b_i^* + \mu_{i-i}b_{i-i}^*| \geq c_3 |b_{i-i}^*|^2 \quad (1 < i \leq n) \tag{3.8}
\]

where

\[
b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{ij}b_j^* \quad (1 \leq i \leq n), \tag{3.9}
\]

\[
\mu_{ij} = \frac{(b_i, b_j^*)}{(b_j^*, b_j^*)} \quad (1 \leq j < i \leq n), \tag{3.10}
\]

and \( c_3 \) is a constant strictly between \( \frac{1}{4} \) and 1, and \( (, \) denotes the ordinary inner product in \( \mathbb{R}^n \).

This is the Gram-Schmidt orthogonalization process. Notice that \( b_i^* \) is the projection of \( b_i \) on the orthogonal complement of \( \sum_{j=1}^{i-1} \mathbb{R}b_j \), and that \( \sum_{j=1}^{i-1} \mathbb{R}b_j = \sum_{j=1}^{i-1} \mathbb{R}b_j^* \), for \( 1 \leq i \leq n \); and \( \mu_{ij} \), for \( 1 \leq j < i \leq n \) are real numbers. Hence \( b_1^*, \ldots, b_n^* \) is an orthogonal basis of \( \mathbb{R}^n \), and \( b_1, \ldots, b_n \), is a reduced basis for a lattice \( (\mathcal{L}) \), if (3.7),(3.8) are satisfied.

Before, we discuss the algorithm, we would like to relate the reduction-processes I and II by showing that they will reduce a given basis in the same sense. We describe it in proposition 1.

Proposition 1. Let \( b_1, \ldots, b_n \) be a reduced basis for a lattice \( (\mathcal{L}) \) in \( \mathbb{R}^n \), let \( b_1^*, \ldots, b_n^* \) be defined as in (3.9). Then we have:

\[
|b_j|^2 \leq 2^{-1}|b_i^*|^2 \quad \text{for} \quad 1 \leq j < i \leq n. \tag{3.11}
\]
\[ d(\mathcal{L}) \leq \prod_{i=1}^{n} |b_i| \leq 2^{n(n-1)} d(\mathcal{L}). \] (3.12)

**Remark:** If \( c_3 \) in (3.8) is \( \frac{3}{4} \), then (3.11),(3.12) are in the form above. If we replace \( c_3 \) by \( y \), with \( 0 < y < 1 \), then the power of 2 appearing in (3.11),(3.12) must be replaced by the same power of \( 4/(4y-1) \).

**Proof:** From (3.7), (3.8), then

\[ |b_i|^2 \geq \left( \frac{3}{4} - \mu_{n-1}^2 \right) |b_{i-1}|^2 \geq \frac{1}{2} |b_{i-1}|^2. \]

or we also can write it as

\[ |b_i|^2 \geq 2^{i-1} |b_{i-1}|^2, \]

\[ |b_{i-1}|^2 \leq 2^{i-1} |b_i|^2 = 2^{i-1} |b_i|^2. \] (3.13)

Now, we can prove by induction that

\[ |b_j|^2 \leq 2^{n-j} |b_n|^2 \quad \text{for} \quad 1 \leq j < i \leq n; \] (3.14)

for \( i = 2 \), it is obvious from (3.13); then we assume that

\[ |b_j|^2 \leq 2^{i-j} |b_i|^2 \quad \text{for} \quad 1 \leq j < i \leq n - 1. \] (3.15)

and from (3.13), letting \( i = n \), we have

\[ |b_{n-1}|^2 \leq 2^n |b_n|^2. \]

Substituting this into (3.15), for \( i = n - 1 \), we get

\[ |b_j|^2 \leq 2^{n-j} |b_n|^2 = 2^{n-j} |b_n|^2, \quad \text{for} \quad 1 \leq j \leq n. \]
If we substitute each \( i = 2, \ldots, n - 2 \) in \((3.13)\) and \((3.15)\), we can prove \((3.14)\).

From \((3.9)\):

\[
b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{ij} b_j^*.
\]

and

\[
|b_i|^2 = |b_i^*|^2 + \sum_{j=1}^{i-1} \mu_{ij}^2 |b_j|^2.
\]

then from \((3.14)\) and \(|\mu_{ij}| \leq \frac{1}{2}\) we get

\[
|b_i|^2 \leq |b_i^*|^2 + \sum_{j=1}^{i-1} \left(\frac{1}{4} (2^{i-2}) |b_j|^2\right)
\]

\[
\leq (1 + \frac{1}{4} (2^{i-2})) |b_i^*|^2
\]

\[
\leq (2^{i-1}) |b_i^*|^2.
\]

It follows that \(|b_j|^2 \leq 2^{j-1} |b_j^*|^2\). From \((3.14)\)

\[
|b_j|^2 \leq 2^{j-1} 2^{j-1} |b_j^*|^2 = 2^{2j-2} |b_j^*|^2.
\]

for \(1 \leq j \leq n\). This finishes the proof of \((3.11)\).

From the determinant of \( \mathbf{L} \), \(d(\mathbf{L}) = |\text{det}(b_1, \ldots, b_n)|\) and from \((3.9)\), \(b_i = b_i^* + \sum_{j=1}^{i-1} \mu_{ij} b_j^*\).

then each column of \( b_i \) is a linear combination of the previous column of \( b_i \). Thus \( \text{det}(\mathbf{L}) = \text{det}(b_1, \ldots, b_n) = \text{det}(b_1^*, \ldots, b_n^*) \). As before, we can write \(d(\mathbf{L})^2 = |\text{det}(b_i, b_j)|_{1 \leq i, j \leq n} \) and since the \( b_j^* \) are pairwise orthogonal, then

\[
d(\mathbf{L}) = \prod_{i=1}^{n} |b_i^*|.
\]

From \(|b_i^*| \leq |b_i|\) and \(|b_i| \leq 2^{\frac{1-1}{2}} |b_i^*|\), then \(\prod_{i=1}^{n} |b_i| \leq 2^{\frac{n-1}{2} \frac{1}{2}} \prod_{i=1}^{n} |b_i^*|\). We now obtain \((3.12)\).

\textbf{Note}: From \((3.12)\), we can see that a reduced basis for The Reduction Process II is also reduced in the same sense as a reduced basis in The Reduction Process I.
The Algorithm for Reduction Process II

Step 1 Compute \( b_i^* \) \((1 \leq i \leq n)\) and \( \mu_j, \) \((1 \leq j < i \leq n)\) from (3.9), (3.10). Use \( c_3 \) as \( 3/4 \).

Step 2 Check whether \( b_i^* \) \((1 \leq i \leq n)\) and \( \mu_j, \) \((1 \leq j < i \leq n)\) satisfy the condition (3.7), (3.8) with \( c_3 = 3/4 \). Let \( k \) be the current subscript and \( k \in \{1, \ldots, n+1\} \). Thus (3.7), (3.8) are

\[
|\mu_j| \leq \frac{1}{2} \quad \text{for} \quad 1 \leq j < k.
\]

(3.16)

\[
|b_i^* + \mu_{i+1} b_{i+1}^*| \geq \frac{3}{4} |b_i^*|, \quad \text{for} \quad 1 < i < k.
\]

(3.17)

If (3.16), (3.17) are satisfied by \( b_1, \ldots, b_k \), then \( b_1, \ldots, b_k \) is a reduced basis of \( \mathcal{L}' \), where \( \mathcal{L}' \) is a lattice in \( \mathbb{R}^n \) and if \( k = n+1 \), then the basis \( b_1, \ldots, b_n \) is a reduced basis for \( \mathcal{L} \in \mathbb{R}^n \). The algorithm terminates.

We begin \( k \) with 2, and for \( k \leq n \), we first achieve that

\[
|\mu_{k-1}| \leq \frac{1}{2}, \quad \text{if} \quad k > 1.
\]

(3.18)

If this doesn't hold, let \( r \) be the nearest integer to \( \mu_{k-1} \) and replace \( b_k \) by \( b_k - rb_{k-1} \). The numbers \( \mu_j, \) with \( j < k - 1 \) are replaced by \( \mu_j - r \mu_{k-1} \), since

\[
\frac{(b_k, b_j^*)}{(b_j^*, b_j^*)} \quad \text{is replaced by} \quad \frac{(b_k - rb_{k-1}, b_j^*)}{(b_j^*, b_j^*)}.
\]

which is

\[
\frac{(b_k, b_j^*)} - \frac{r(b_{k-1}, b_j^*)} = \mu_{j-1} - r \mu_{k-1}.
\]

Moreover, \( \mu_{k-1} \) becomes \( \mu_{k-1} - r \), since \( \mu_{k-1} = \frac{(b_k, b_{k-1}^*)}{(b_{k-1}^*, b_{k-1}^*)} \) is replaced by \( \frac{(b_k - rb_{k-1}, b_{k-1}^*)}{(b_{k-1}^*, b_{k-1}^*)} \), which is \( \mu_{k-1} - r \). The other \( \mu_j, (i \neq k) \) and \( b_i^* (i < k) \) are unchanged.
Case of $i = k$

$$b^*_k = b_k - \sum_{j=1}^{k-1} \mu_{kj} b^*_j.$$  

$b^*_k$ is replaced by $(b_k - rb_{k-1}) - \sum_{j=1}^{k-2} (\mu_{kj} - r\mu_{k-1,j}) b^*_j - (\mu_{k,k-1} - r)b^*_{k-1} = (b_k - \sum_{j=1}^{k-1} \mu_{kj} b^*_j) - r(b_{k-1} - \sum_{j=1}^{k-2} \mu_{k-1,j} b^*_j) + rb^*_{k-1} = b^*_k - rb^*_{k-1} + rb^*_{k-1} = b^*_k.$

Thus, it is clear that $b^*_i (i > k)$ remain unchanged.

After all these changes, we have a new basis $b_1, \ldots, b_n$ such that $b_k$ is replaced by $b_k - rb_{k-1}$ and all other $b_i (i \neq k)$ are unchanged, and (3.16) holds. Then we distinguish 2 cases.

Case 1 Suppose $k \geq 2$ and $|b^*_k + \mu_{k,k-1} b^*_{k-1}|^2 < \frac{1}{2} |b^*_{k-1}|^2$ In this case

a) interchange $b_{k-1}$ and $b_k$ leave the other $b_i$ the same.

b) the vector $b^*_{k-1}$ and $b^*_k$ and the numbers $\mu_{k,k-1}, \mu_{k-1,j}, \mu_{j,k}, \mu_{k,k-1}, \mu_{i,j}$ for $j < k - 1$ and $i > k$ have now to be changed.

The following are the formulas for changing all elements in a), b) The proof for these formulas will be given after finishing the description of this algorithm.

Let $c_1, c_2, \ldots, c_n$ denote the vectors and the numbers that will replace $b_i, b^*_i$ and $\mu_{i,j}$ respectively.

The new basis $c_1, \ldots, c_n$ is given by

$$c_{k-1} = b_{k-1}, \quad c_k = b_k, \quad c_i = b_i \quad \text{for} \quad i \neq k - 1, k.$$

Let $c^*_i$ be the projection of $c_i$ such that $c^*_i = c_i - \sum_{j=1}^{k-1} \mu_{ij} c^*_j$.

**Formula 1:**

$$c^*_{k-1} = b^*_k + \mu_{k,k-1} b^*_{k-1}.$$
Formula 2 \[ v_{k+1} = \mu_{k+1} - t_k v_{k+1} + \mu_{k+1} t_k v_{k+1} \]

Formula 3 \[ c_k = \sum_{i=1}^{k-1} |c_i|^2 \]

Formula 4 \[ t_{i+1} = \mu_{i+1} - \mu_i \]

and

\[ t_i = \mu_i - |u_{i-1} - \mu_i u_{i-1}| \]

Formula 5 \[ \sum_{i=1}^{k-1} t_i = \mu_i \]

and

\[ \sum_{i=1}^{k-1} v_i = \mu_i \]

Formulas 6 \[ \text{if } 1 \leq j < i \leq n \text{ and } \{i, j\} \cap \{k-1, k\} = \emptyset \text{ then} \]

\[ v_i = \mu_i \]

\begin{itemize}
  \item \textbf{Remarks:} For \( j > k-1 \) and \( i < k \) there doesn't exist values of \( \mu_{k-1}, \mu_k, \mu_{i-1}, \mu_i \) there is a value for \( \mu_i \) only if \( 1 \leq s < r \leq n \)
\end{itemize}

After these changes have been made we replace \( k \) by \( k-1 \) Then we are in the situation described by (3.11) (3.12) we proceed with the algorithm from there

\begin{itemize}
  \item \textbf{Case 2:} Suppose that \( k = 1 \) or \( |b_1^* + \mu_{k-1} b_{k-1}^*|^2 \leq \frac{3}{4} |b_{k-1}^*|^2 \) in this case we first achieve that
  \[ |\mu_i| \leq \frac{1}{2} \text{ for } 1 \leq j \leq k - 1 \quad (3.19) \]
\end{itemize}
If (3.19) doesn't hold, let \( l \) be the largest index < \( k \) with \( |\mu_{kl}| > \frac{1}{2} \), let \( r \) be the integer nearest to \( \mu_{kl} \), and replace \( b_k \) by \( b_k - rb_l \). The numbers \( \mu_{kj} \) with \( j < l \) are then replaced by \( \mu_{kj} - r\mu_{kl} \), and \( \mu_{kl} \) by \( \mu_{kl} - r \); the other \( \mu_{ij} \) and all \( b_i^* \) are unchanged. This is repeated until (3.19) holds.

Next, we replace \( k \) by \( k + 1 \), then we are in the situation described by (3.11), (3.12); and we proceed with the algorithm from there.

Notice that in the case \( k = 1 \), we have done no more than replace \( k \) by 2. This finishes the description of the algorithm.

**Proof of the formulas used in Case 1**

Let \( b_1, \ldots, b_n \) be the current basis and \( b_i^*, \mu_{ij} \) be as in (3.9), (3.10); \( k \) be the current subscript as defined in the algorithm. By \( c_i, c_i^* \) and \( v_i \), we denote the vectors and numbers that will replace \( b_i, b_i^* \) and \( \mu_{ij} \) respectively. Then the new basis is given by

\[
c_{k-1} = b_k, \quad c_k = b_{k-1}, \quad c_i = b_i, \quad \text{for} \quad i \neq k-1, k.
\]

**Formula 1**: to prove \( c_{k-1}^* = b_k^* + \mu_{kk-1} b_{k-1}^* \).

**Proof**

Since \( c_{k-1}^* \) is the projection of \( b_k \) on the orthogonal complement of \( \sum_{j=1}^{k-2} \mathbb{R} b_j \), and \( b_k^* = b_k - \sum_{j=1}^{k-2} \mu_{kj} b_j^* \), then

\[
c_{k-1}^* = b_k - \sum_{j=1}^{k-2} \mu_{kj} b_j^*.
\]

and

\[
b_k^* = b_k - \sum_{j=1}^{k-2} \mu_{kj} b_j^* - \mu_{kk-1} b_{k-1}^*.
\]

Hence,

\[
c_{k-1}^* = b_k^* + \mu_{kk-1} b_{k-1}^*.
\]
Formulas 2.5.5.6: to prove

\[ v_{k+1} = u_{k+1} |b_{k+1}^*|^2 / |c_k^*|^2. \]

\[ c_k^* = b_{k+1}^* - v_{k+1} c_k^*. \]

\[ v_{k+1} = \mu_{k+1}, \quad v_{k+1} = \mu_{k-1}, \quad \text{if} \quad (1 \leq j < k - 1). \]

and

\[ v_{ij} = \mu_{ij}; \quad \text{if} \quad (i, j) \cap \{k - 1, k\} = \emptyset, \quad 1 \leq j < i \leq n. \]

Proof.

From

\[ c_k^* = c_k - \sum_{j=1}^{k-1} v_{kj} c_j^*. \tag{3.20} \]

and \( c_k^* = b_k^* \) for \( k \neq k - 1, k \).

\[ v_{kj} = \frac{(c_k, c_j^*)}{(c_j^*, c_j^*)} = \frac{(b_{k-1}, b_j^*)}{(b_j^*, b_j^*)}; \quad j \neq k - 1, k. \]

and

\[ v_{kj} = \mu_{k-1}. \]

Similarly,

\[ v_{k-1,j} = \mu_{k-1}, \]

\[ v_{k+1,j} = \mu_{k-1}, \quad \text{for} \quad 1 \leq j < k - 1. \]

and

\[ v_{ij} = \mu_{ij}; \quad 1 \leq j < i \leq n, \quad (i, j) \cap \{k - 1, k\} = \emptyset. \]
These prove formula 5 and formula 6. From (3.20), we have

$$c_k^* = c_k - \sum_{j=1}^{k-1} v_{kj} c_j^* = v_{kk-1} c_{k-1}^*,$$

$$= b_{k-1} - \sum_{j=1}^{k-1} \mu_{k-1} c_j^* = v_{kk-1} c_{k-1}^*.$$

then

$$c_k^* = b_{k-1}^* - v_{kk-1} c_{k-1}^*. \quad (3.21)$$

From the above equation, we can obtain $v_{kk-1}$ by taking the projection of $b_{k-1}^*$ on the orthogonal complement of $\mathbb{F} c_{k-1}^*$. This leads to

$$v_{kk-1} = \frac{(b_{k-1}^*, c_{k-1}^*)}{(c_{k-1}^*, c_{k-1}^*)},$$

$$= \frac{(b_{k-1}^*, (b_{k-1}^* + \mu_{k-1} b_{k-1}^*)}{|c_{k-1}^*|^2}.$$

By using the formula 1, and $b_{k-1}^*$ is perpendicular to $b_{k-1}^*$, then

$$v_{kk-1} = \mu_{k-1} |b_{k-1}^*|^2 / |c_{k-1}^*|^2.$$

This finishes the proofs for formula 2, formula 3.

**Formula 4**: for $i > k$,

$$v_{ik} = \mu_{ik-1} v_{k+1} + \mu_{ik} |b_{k}^*|^2 / |c_{k-1}^*|^2.$$

and

$$v_{ik} = \mu_{ik-1} - \mu_{ik-1} \mu_{kk-1}.$$

**Proof**

Since $v_{ik-1} = \frac{(b_i, c_{k-1})}{(c_{k-1}^*, c_{k-1}^*)}$, and $b_i = c_i$ for $i \neq k - 1, k$; then

$$v_{ik} = \frac{(b_i, c_{k-1}^*)}{|c_{k-1}^*|^2}.$$
From formula 1, formula 2:

\[ v_{ik-1} = \frac{(b_ia_i + \mu_{kk-1}b_{k-1}^*)}{|c_{ik-1}^*|^2}, \]

\[ = \frac{(b_ia_i + \mu_{kk-1}b_{k-1}^*)}{|c_{ik-1}^*|^2} = \frac{(b_ia_i + \mu_{kk-1}b_{k-1}^*)}{|c_{ik-1}^*|^2}, \]

\[ = \mu_{ik} \frac{|b_i^*|^2}{|c_{ik-1}^*|^2} + \mu_{kk-1} \frac{|b_{k-1}^*|^2}{|c_{ik-1}^*|^2}. \]

Hence,

\[ v_{ik-1} = \mu_{ik} \frac{|b_i^*|^2}{|c_{ik-1}^*|^2} + \mu_{ik-1} v_{k-1}; \quad \text{for } i > k. \]

In the same way, we can obtain the second equation such that \( v_{ik} = \mu_{ik-1} - \mu_{ik} \mu_{kk-1} \); for \( i > k \).

This finishes the proof.

We shall include the entire algorithm of the reduction process II in Figure 1.2. Let \( E_i = |b_i^*|^2 \).

and \( k \) be a current subscript.
The entire Algorithm for The Reduction Process II

(2a)

For $i = 1$ to $n$ do

begin

$b_i^* = b_i$

For $j = 1$ to $i - 1$ do

begin

$\mu_j = (b_j, b_i^*) / E_j$

$b_j^* = b_j^* - \mu_j b_i^*$

end:

$E_i := (b_i^*, b_i^*)$

end.

$k = 2$

(1)

Perform (a) for $l = k - 1$

If $E_k < (3/4 - \mu_{k-1}^*) E_{k-1}$ goto (3).

Perform (a) for $l = k, k - 2, k - 3, \ldots, 1$

If $k = n$, Terminate.

$k := k + 1$

goto (1).
(2)

\[ \begin{align*}
\mu &= \mu_{k-1} \\
E &= E_k + \mu^2 E_{k-1} \\
\mu_{k+1} &= (\mu E_{k-1}) / E \\
E_k &= (E_{k-1} E_k) / E \\
E_{k-1} &= E
\end{align*} \]

\[ \begin{pmatrix} b_{k-1} \\ b_k \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_k \\ b_{k-1} \end{pmatrix} \]

For \( i = 1, 2, \ldots, k-2 \) do

\[ \begin{pmatrix} \mu_{k-1} \\ \mu_k \end{pmatrix} = \begin{pmatrix} \mu_k \\ \mu_{k-1} \end{pmatrix} \]

For \( i = k+1, \ldots, n \) do

\[ \begin{pmatrix} \mu_{k-1} \\ \mu_k \end{pmatrix} = \begin{pmatrix} 1 & \mu_{k-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -\mu \end{pmatrix} \begin{pmatrix} \mu_{k-1} \\ \mu_k \end{pmatrix} \]

If \( k > 2 \) Then

\[ \begin{align*}
k &= k - 1 \\
goto \text{(1)}
\end{align*} \]
\[ (*) \]

If \(|\mu_{k_{1}}| > 1/2\) Then

\[ r = \text{integer nearest to } \mu_{k_{1}}. \]

\[ b_{k_{1}} = b_{k_{1}} - rb_{k_{1}}. \]

For \( j = 1, 2, \ldots, l - 1 \) do

\[ \mu_{k_{j}} = \mu_{k_{j}} - \mu_{k_{j}}. \]

\[ \mu_{k_{l}} = \mu_{k_{l}} - r. \]

**Figure 3-1:**

**Remark:** After (**), it is not necessary to keep track of the vectors \( b_{k}^{*} \). It suffices to keep track of the numbers \( |b_{k}^{*}|^2 \), in addition to \( \mu_{k} \) and the vector \( b_{k} \). Notice that \( |c_{k-1}^{*}|^2 = (|b_{k-1}^{*}|^2 |b_{k}^{*}|^2) / |c_{k-1}^{*}|^2 \) (from formula 2.3), and \( |b_{k}^{*} + \mu_{k-1} b_{k-1}^{*}|^2 = |b_{k}^{*}|^2 + \mu_{k-1}^2 |b_{k-1}^{*}|^2 \), since \( b_{k}^{*} \) is perpendicular to \( b_{k-1}^{*} \).
Complexity of the algorithm

Let $B \in \mathbb{R}$ and $B \geq 2$ with the basis $b_1, \ldots, b_n$ for $L$ such that $|b_i|^2 \leq B$ for $1 \leq i \leq n$.

Using Figure 3.1 to analyse the running time, the basic operation is the number of times passing through (1) and (2). In (2) the value of $k$ is decreased by 1, and in (1), it is increased by 1. Initially, we have $k = 2$, $k \leq n + 1$ throughout the algorithm, therefore the number of times that we pass through (1) is at most $(n - 1)$ more than the number of times that we pass through (2).

If we are in the situation of (2), implies that $E_k < (3/4 - \mu_{k-1}^2)E_{k-1}$, where $E_k = |b_k|^2$. The question is how many times we have to go through (2) to get into the situation that $E_k > (3/4 - \mu_{k-1}^2)E_{k-1}$ for all $1 \leq k \leq n$.

In order to get an upper bound for the number of times passing through (2), we introduce the quantities

$$d_i = \text{det}([b_j, b_i])_{1 \leq j, i \leq n} \quad (3.22)$$

Since $b_i = b_i^* + \sum_{j=1}^{i-1} \mu_j b_j^*$, and $b_1^*, \ldots, b_n^*$ is an orthogonal basis of $L^*$, then $d_i = \text{det}([b_i^*, b_i^*])$ for $1 \leq j, i \leq n$.

$$d_i = \prod_{j=1}^{i-1} |b_j|^2 \quad (3.23)$$

Hence the $d_i$ are positive real numbers since $d(L) = \prod_{j=1}^{n} |b_j^*|^2$ it follows that $d_n = d(L)^3$.

In (2), if $|b_k^* + \mu_{k-1} b_{k-1}^*|^2 \leq 3/4 |b_{k-1}^*|^2$, then we replace $b_{k-1}^*$ by $b_k^* + \mu_{k-1} b_{k-1}^*$ (from Formula 1). Then the new $|b_{k-1}^*|^2$ is at most $3/4$ times the old one. From (3.23), the number $d_{k-1}$ is reduced by a factor of at most $3/4$, whereas the other $d_i, 1 \leq i \leq k - 1$ are unchanged. From Formulas 2, 3, it follows that the product of norms for the new $b_k^*, b_{k-1}^*$ is unchanged from the product
of the old ones, hence it implies that \( d_i, i > k - 1 \) are unchanged. If we let

\[
D = \prod_{i=1}^{k-1} d_i,
\]

(3.24)

then the number \( D \) is changed if some \( d_i \) is changed. Hence, \( D \) is reduced by a factor at most \( 3/4 \).

If we are in (2) with the current dimension \( k \), after replacing \( |b_{k-1}^-|^2 \) with a value less than \( 3/4 \) times \( |b_{k-1}^-|^2 \) and if \( k > 2 \), then we are in the same situation as described above, which is checked for \( |b_k^- + \mu_k b_{k-1}^-|^2 \leq 3/4 |b_{k-1}^-|^2 \) for \( k \) replaced by \( k - 1, 2 \leq k \leq n \). In the worst case, if we let \( r \) be the number of times passing through (2): If we start with the sequence

\[
|b_1^-|^2, \ldots, |b_k^-|^2,
\]

where \( |b_1^-|^2 \) is a maximum norm of this sequence, and \( |b_k^-|^2 \) is a minimum norm of this sequence. We will generate the sequence of

\[
(3/4)^r |b_1^-|^2, (3/4)^{r-1} |b_1^-|^2, \ldots, |b_1^-|^2,
\]

and \( |b_1^-|^2 \) is bounded by a value of \( d_k \). The upper bound of \( (3/4)^r |b_1^-|^2 \) is the value of the minimum norm of the started sequence which is \( |b_k^-|^2 \). This sequence satisfies the condition \( |b_k^- + \mu_{i-1} b_{i-1}^-|^2 > 3/4 |b_{i-1}^-|^2 \), \( i < k \). If let \( p = (3/4)^r |b_1^-|^2 \), then \( p \leq |b_k^-|^2 \). From (3.25),

\[
p, \left(\frac{4}{3}\right)^r p, \left(\frac{4}{3}\right)^r p, \ldots, \left(\frac{4}{3}\right)^r p.
\]

Hence, \( (\frac{4}{3})^r p \leq d_k \), then

\[
\frac{\log d_k}{\log(4/3)} \leq \frac{\log(d_k)}{p} \leq \log(d_{k-1}).
\]
The situation described above can happen for every \( 2 \leq k \leq n \). Hence, the number of times that the algorithm goes through (2) is
\[
\log d_1 + \cdots + \log d_{n-1} = \log \prod_{i=1}^{n-1} d_i = \log D.
\]

The bound \( B \geq |b_i|^3 \) implies that \( B' \geq \prod_{j=1}^n |b_j|^3 \). By (3.12), \( \prod_{j=1}^n |b_j|^3 \geq d(L)^3 \geq d_n \); this implies that \( \prod_{j=1}^n |b_j|^3 \geq d \). Hence, \( B' \geq d \), and \( \prod_{i=1}^{n-1} B' \geq \prod_{i=1}^{n-1} d_i \); then by (3.25), \( D \leq B^{(n-1)/3} \). Thus, an upper bound for the number of times passing through (2) is \( \log B^{(n-1)/3} \), which is \( O(n^3 \log B) \).

In Case 1, the number of times passing through is \((n-1)\) more than Case 2. Then an upper bound for the number of times passing through (1) is \( O((n \log B) + (n-1)) \) which is \( O(n^2 \log B) \). In (1), to perform (*) we need \( O(n) \) arithmetic operations (we mean an addition, subtraction, multiplication, division).

We perform (*) for the values of \( L \) from \( k - 2 \) to \( 1 \), then we need \( O(n^3) \) of the arithmetic operations for (1). Hence, the overall performance for (1) is the product of \( O(n^2) \) and \( O(n^3 \log B) \) which is
\[
O(n^4 \log B).
\]

In (2), we need \( O(n) \) to perform the \( \mu \)-value then the overall performance for (2) is the product of \( O(n) \) and \( O(n^3 \log B) \) which is \( O(n^4 \log B) \). In (**), we need \( O(n^3) \) arithmetic operations.

From the above information, we can get the overall performance for the whole algorithm by summation of the time required in (**), (1), and (2), which is \( O(n^2) + O(n^3 \log B) + O(n^4 \log B) \). Thus, the complexity of this algorithm is \( O(n^4 \log B) \), which is bounded by a polynomial function in the length of input. ( \( B \) is an upper bound of the input basis.)
3.3 Implementation of The Reduction Process

The following are the implementations of The Reduction Process I II. Both implementations are used on the Digital Professional 350 microcomputer, and they are implemented in the USCD p-system. The structure of data is the following:

```
| position | thisvector | nextvector |
```

![Diagram of data structure](image)

Figure 3.2:

The procedure `GetBasis` is the procedure to read-in an input, and also construct those input as a linked list as Figure 3.2. This procedure is using in both implementations, we only present this procedure in Reduction Process I. This procedure also checks for some possible errors, for e.g., input as a character instead of a number.

Some limitations for both algorithms:

a) input are vectors of m elements, and n vectors where m, n, can not be greater than 10.
This is because of the limitation of memory in the microcomputer. Those $m$ vectors are linearly independent.

b) If there occurs an error such that Stack overflow, it means that the computer has used all the memory. You should turn the computer off and turn it on again and re-input (the computer will proceed the garbage collection and also re-initialize the system). If you get an error message again this time, it means that your problem take too many iterations, and the computer's memory is too small for your problem.

c) There is no restriction on $m$. $m$ so that $m$ can be greater than or less than $n$. 
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Implementation of The Reduction Process I

Program

Global Variable

Const  
  ItemMax = 6.

Type  
  ArrayBoolean = Array 1 ItemMax of boolean.
  ItemArray = array 1 ItemMax of real
  Matrix = array 1 ItemMax 1 ItemMatrix of integer
  Item = array 1 ItemMax of integer
  PToMatrix = Matrix
  ArrayOfMatrix = Array 1 ItemMax PToMatrix
  ElementPointer = ListElement.

ListElement =
  record
    Data real.
    Next ElementPointer.
  end.

HeadPointer = HeadElement.

HeadElement =
  record
    Position : integer;
    ThisVector : ElementPointer;
    NextVector : HeadPointer;
  end.

Var

OutData : text;
IsSave : ArrayBoolean;
NumElement, Dimension, Dim, IndexOfMatrix, Count, Index, AccuCount : integer;
HeadOfBasis, AnotherHead, OriginalBasis : HeadPointer;
RightBasis, IsRecursive, FirstRound, IsOkay : boolean;
Aconstant, Pconstant, Determinant : real;
ArrayOfMatrix : ArrayOfMatrix;
CoefMatrix PirToMatrix
Norm ItemArray
NValue Item
Command char
Segment Procedure GetBasis (Var HeadOfBasis, HeadPointer, Var IsOkay boolean)

Var Dimension: integer;

Var lIndex: integer;
Okay, Negative boolean;
Member ArrayNorm:
Count: integer;
Base, Temp1, HeadPointer:
Vector, Temp2: ElementPointer;

Function ParseGroup boolean;
Const Eos = '';
Var S: Prompt: string;
P: Index: integer;

Procedure ParseError (Msg: integer):
begin
    writeln('': length(Prompt), ': , ', Msg);
    exit(ParseGroup);
end; {ParseError}

Procedure SkipSpaces;
begin
    while S[P] = '' do
        P := P + 1;
end; {SkipSpaces}

Function ParseNum: real;
Var I: integer;
N, D: real;
begin
    SkipSpaces;
    If not (S[P] in ['0'..'9', '+', '-']) then
        ParseError ('Expecting number ');

\[ I = 0; \]
\[ N = 0; \]
\[ D \leftarrow 0; \]
\[ \text{if } S[P] = \'\cdot\' \text{ then} \]
\[ \begin{array}{l}
\text{begin} \\
\quad \text{Negative} := \text{true}; \\
\quad P := P + 1; \\
\text{end} \\
\text{else} \\
\quad \text{Negative} := \text{false}; \\
\text{if } (S[P] \neq \cdot) \text{ then} \\
\quad \begin{array}{l}
\quad \text{repeat} \\
\quad \quad N := N \ast 10 + \text{ord}(S[P]) - \text{ord}(\'0\'); \\
\quad \quad P := P + 1; \\
\quad \text{until } (S[P] \text{ in } \[0, \ldots, 9\] \text{ or } (S[P] = \cdot)); \\
\quad \text{if } (S[P] = \cdot) \text{ then} \\
\quad \quad \begin{array}{l}
\quad \quad P := P' + 1; \\
\quad \quad \text{while } S[P] \text{ in } \[0, \ldots, 9\] \text{ do} \\
\quad \quad \quad \begin{array}{l}
\quad \quad \quad \begin{array}{l}
\quad \quad \quad \quad I := I + 1; \\
\quad \quad \quad \quad D := D \ast 10 + \text{ord}(S[P]) - \text{ord}(\'0\'); \\
\quad \quad \quad \quad P := P + 1; \\
\quad \quad \quad \end{array} \\
\quad \quad \quad \end{array} \\
\quad \quad \end{array} \\
\quad \text{if not } (S[P] \text{ in } \[0, \ldots, 9, \ldots, \ldots, \cdot\cdot\cdot\]) \text{ then} \\
\quad \quad \text{ParseError ('Expecting number');} \\
\quad \quad N := N \ast \text{rpeofen}(I)); \\
\quad \text{if Negative then} \\
\quad \quad \quad \text{ParseNum} := -(N) \\
\quad \text{else} \\
\end{array} \]
ParseNum = N.

end \{ParseNum\}

\begin{align*}
\text{begin} \{\text{ParseGroup}\} & \\
\text{Index} & = 1, \\
\text{Prompt} & = \text{'Enter bias please'} \\
\text{write}(\text{Prompt}) & : \\
\text{readln}(S) & : \\
S & = \text{Concat}(S, \text{Eos}). \\
\text{ParseGroup} & = \text{false}. \\
P & = 1. \\
\text{SkipSpaces} & : \\
\text{if} S[P] <> \text{Eos} \text{then} & \\
\text{begin} & \\
\text{Member}[\text{Index}] & = \text{ParseNum}. \\
\text{SkipSpaces} & : \\
\text{while} S[P] <> \text{Eos} \text{do} & \\
\text{if} S[P] <> \text{'}: \text{then} & \\
\text{ParseError} (\text{'Expecting'} \text{')} & \\
\text{else} & \\
\text{begin} & \\
P & := P + 1; \\
\text{Index} & := \text{Index} + 1; \\
\text{Member}[\text{Index}] & := \text{ParseNum}. \\
\text{SkipSpaces} & : \\
\text{end} & \\
\text{end}; \\
\text{while} \text{Index} < \text{NumElement} \text{do} & \\
\text{begin} & \\
\text{Index} & := \text{Index} + 1; \\
\text{Member}[\text{Index}] & := 0; \\
\text{end}; \\
\end{align*}
begin (ctlBase)

If (Okay then

end (ForwBmp)
new(Basis);
with Basis : do
begin
  Position = Count;
  ThisVector = nil;
  NextVector = nil;
end;
Temp1 := NextVector := Basis;
Temp1 := Temp1 := NextVector;
new(Vector);
with Vector : do
begin
  Data := Member[1];
  Next := nil;
end;
Temp1 := ThisVector := Vector;
Temp2 := Temp1 := ThisVector;
for I := 2 to NumElement do begin
  new(Vector);
  with Vector : do
  begin
    Data := Member[I];
    Next := nil;
  end;
end;
else
begin
  writeln('Re-enter your data.');</n  IsOkay := false;
exit(GetBasis);
end:

until (Count = Dimension).
end
else
begin

writeln('Valid dimension is in between 1 and ItemMax.

isOkay = false.

end.
end {setBasis}
Function ZeroMatrix (AMatrix, PirToMatrix, Size, Integer).

Var I, J: Integer.

begin

For I = 1 to Size do

  For J = 1 to Size do

    AMatrix[I, J] := 0;

  ZeroMatrix := AMatrix;

end. {ZeroMatrix}


begin

  IsEmpty := true;

  else

    IsEmpty := false;

  end. {IsEmpty}

Procedure InitArray (var whichArray: ArrayBoolean).

Var

  I: Integer.

begin

  For I = 1 to ItemMax do

    WhichArray[I] := false;

end. {InitArray}
Procedure WriteMyResult (HeadOfBasis : HeadPointer; Norm : ItemArray);

Var
    Temp1 : HeadPointer;
    Temp2 : ElementPointer;
    I : Integer;

begin
    Temp1 := HeadOfBasis;
    I := 0;
    Repeat
        I := I + 1;
        Temp1 := Temp1^.NextVector;
        Write (OutData, '('.
        Temp := Temp1^.ThisVector;
        While (Temp^.Next <> nil) do
            begin
                write (OutData, Temp^.Data : 9:2, ' ');
                Temp2 := Temp^.Next;
            end;
            write (OutData, Temp2^.Data : 9:2, '
            writeln (OutData);
            Until (Temp1^.NextVector = nil);
            writeln (OutData, '=================================
            writeln (OutData);
        end; {WriteMyResult}
Function NewHead (Size : NumElement; Integer) HeadPointer

Var
    Base : Temp1 HeadPointer;
    Vector : Temp2 ElementPointer;
    Count : I Integer;

begin
    New (Base);
    With Base do
    begin
        Position = 0;
        ThisVector = nil;
        NextVector = nil;
    end;
    NewHead = Base;
    Temp1 = Base;
    Count = 0;
    Repeat
        Count = Count + 1;
        New (Base);
        With Base do
        begin
            Position = Count;
            ThisVector = nil;
            NextVector = nil;
        end;
        Temp1 | NextVector = Base;
        Temp1 = Temp1 | NextVector;
        New (Vector);
        With Vector do
        begin
            Data := 0;
        end;
    until Count >= Size;
end.
Program FindNorm ( Dim : Integer; Var Norm : ItemArray; HeadOfBasis : HeadPointer )

Var
  Templ : HeadPointer;
  Temp2 : ElementPointer;
  Dum : ItemArray;
  I, J : Integer;

Procedure Initialize;

Var
  I : Integer;

begin
  For I = 1 to Dim do
    Dum[I] = 0;
end;

begin
  Initialize;
  Templ := HeadOfBasis;
  I := 1;
  While (Templ^.NextVector <> nil) do
    begin
      Templ := Templ^.NextVector;
      Temp2 := Temp2^.ThisVector;
      Dum[I] := Dum[I] + Sqr(Temp2^.Data);
      Repeat
        Temp2 := Temp2^.Next;
        Dum[I] := Dum[I] + Sqr(Temp2^.Data);
      Until (Temp2^.Next = nil);
      Norm[I] := Sqrt[I];
      I := I + 1;
    end;
end. {FindNorm}
Procedure GetMinMaxAndReorder (Dim : integer, Norm : ItemArray, HeadOfBasis : HeadPointer, MinNorm : real)

Var
   Temp1, Temp2, Temp3 : HeadOfPointer
   Location : integer

begin
   MinNorm = Norm[1]
   Location = HeadOfBasis.NextVector.Position
   For I = 2 to Dim do
      If (Norm[I] < MinNorm) then
         begin
            Location = I
            MinNorm = Norm[I]
         end.
      Temp2 = HeadOfBasis.
   For I = 1 to Location do
      begin
         Temp3 = Temp2.
         Temp2 = Temp2.NextVector
         Temp1 = Temp2.NextVector.
      end.
   If (Temp2.Position <> HeadOfBasis.NextVector.Position) then
      begin
         If (Temp3.Position = HeadOfBasis.NextVector.Position) then
            begin
               HeadOfBasis.NextVector = Temp2
               Temp3.NextVector = Temp1
            end.
         else
            begin
            end.
      end.
\[ Temp2 \mid \text{NextVector} := \text{Temp3} \mid \text{NextVector} \mid \text{NextVector} \\]
\[ \text{Temp3} \mid \text{NextVector} \mid \text{NextVector} := \text{Temp1} \\]
\[ \text{HeadOfBasis} \mid \text{NextVector} := \text{Temp2} \\]

end

end. \{(GetMinNormAndReNumber\} \)
begin
  Top := Top + 1;
  AStack[Top] := Amatrix;
end. {PushMatrix}

begin
  If Not IsEmpty (Top) then
  begin
    Amatrix := AStack[Top];
    Top := Top - 1;
  end
  else
  begin
    writeln (OutData, '....Stack is empty....');
    Exit(Reduction);
  end;
end. {PopMatrix}
Function \( \text{NewHead}(\text{Size}, \text{NumElement}, \text{integer}) \) \( \text{HeadPointer} \)

\[ \text{Var} \]

\( \text{Base}, \text{Temp1}, \text{HeadPointer} \)
\( \text{Vector}, \text{Temp2}, \text{ElementPointer} \)
\( \text{Count}, I, \text{integer} \)

\[ \text{begin} \]

\( \text{New}(\text{Base}) \)
\( \text{With Base} \) do

\[ \text{begin} \]

\( \text{Position} = 0 \)
\( \text{ThisVector} = \text{nil} \)
\( \text{NextVector} = \text{nil} \)

\[ \text{end}; \]

\( \text{NewHead} = \text{Base} \)
\( \text{Temp1} = \text{Base} \)
\( \text{Count} = 0 \)

\[ \text{Repeat} \]

\( \text{Count} = \text{Count} + 1 \)
\( \text{New}(\text{Base}) \)
\( \text{With Base} \) do

\[ \text{begin} \]

\( \text{Position} = \text{Count} \)
\( \text{ThisVector} = \text{nil} \)
\( \text{NextVector} = \text{nil} \)

\[ \text{end}; \]

\( \text{Temp1} \uparrow \text{NextVector} = \text{Base} \)
\( \text{Temp1} = \text{Temp1} \uparrow \text{NextVector} \)
\( \text{New}(\text{Vector}) \)
\( \text{With Vector} \) do

\[ \text{begin} \]

\( \text{Data} := 0 \)

\[ \text{end}; \]

\[ \text{end}; \]

\[ \text{end}; \]

\[ \text{end}; \]
Next := nil;

end;

Temp1 := ThisVector := Vector;
Temp2 := Temp1 := ThisVector;
For I := 2 to NumElement do begin
  New(Vector);
  With Vector do begin
    Data := 0;
    Next := nil;
  end;
  Temp2 := Next := Vector;
  Temp2 := Temp2 := Next;
end:

Until (Count = Size);
end; {NewHead}
Function Copy (Head : HeadPointer; Size : Integer) : HeadPointer;

Var

Ahead, Temp1, Temp2 : HeadPointer;
Temp3, Temp4 : ElementPointer;

begin
Ahead := NewHead (Size, NumElement);
Temp1 := Head;
Temp2 := Ahead;

While Temp1 <> NextVector do
begin
Temp1 := Temp1 | NextVector;
Temp2 := Temp2 | NextVector;
Temp3 := Temp1 | ThisVector;
Temp4 := Temp2 | ThisVector;

While Temp3 <> nildo
begin
Temp4 | Data := Temp3 | Data;
Temp3 := Temp3 | Next;
Temp4 := Temp4 | Next;
end;
end;

Copy := Ahead;
end; {Copy}
Procedure GetProjection (Var AnotherHead : HeadPointer ; Dim : Integer ;
    HeadOfBasis : HeadPointer, Var NValue : Item ) ;

Var

Temp1, Temp2, Temp6 : HeadPointer ;
Temp3, Temp4, Temp6 : ElementPointer ;
Division, InnerProduct, Multiplier : real ;
I : Integer ;

begin

AnotherHead := NewHead (Dim - 1, NumElement ) ;
Temp6 := AnotherHead ;
I := 0 ;
Temp3 := HeadOfBasis | NextVector | ThisVector ;
Division := Sqr (Temp3 | Data) ;
Repeat

    Temp3 := Temp | Next ;
    Divisor := Divisor + sgr (Temp3 | Data) ;
Until (Temp3 | Next = nil) ;
Temp2 := HeadOfBasis | NextVector ;
Repeat

    I := I + 1 ;
    Nvalue[I] := 0 ;
    Temp6 := Temp | NextVector ;
    Temp2 := Temp2 | NextVector ;
    Temp3 := HeadOfBasis | NextVector | ThisVector ;
    Temp4 := Temp2 | ThisVector ;
    InnerProduct := Temp4 | Data * Temp3 | Data ;
    Repeat

        Temp3 := Temp3 | Next ;
        Temp4 := Temp4 | Next ;
        InnerProduct := InnerProduct + (Temp4 | Data * Temp3 | Data)
    Until (Temp4 | Next := nil) ;

end


Multiplier := InnerProduct/Divisor.

If isRecursive then


Temp5 := Temp6 \cdot ThisVector;

Temp3 := HeadOfBasis \cdot NextVector \cdot ThisVector;

Temp4 := Temp2 \cdot ThisVector;

Temp5 \cdot Data := Temp4 \cdot Data - (Multiplier \cdot Temp3 \cdot Data);

Temp4 \cdot Data := Temp4 \cdot Data + NValue[I] \cdot Temp3 \cdot Data.

Repeat

Temp3 := Temp3 \cdot Next;

Temp4 := Temp4 \cdot Next;

Temp5 := Temp5 \cdot Next;

Temp5 := Temp4 \cdot Data - (Temp3 \cdot Data \cdot Multiplier);

Temp4 := Temp4 \cdot Data + (Nvalue[I] \cdot Temp \cdot Data);

Until (Temp4 \cdot Next = nil);

Until (Temp2 \cdot NextVector = Nil);

end; \{GetProjection\}
Function NewBasis (PoppedElement : HeadPointer; TheMatrix : PtrToMatrix; Size : Integer)

    HeadPointer;

Var
    Temp1, Temp4, AnotherHead, Temp6 : HeadPointer;
    Temp2, Temp3, Temp5 : ElementPointer;
    I, J : Integer;

begin
    AnotherHead = NewHead (Size, NumElement);
    Temp1 := PoppedElement;
    Temp6 := AnotherHead;
    For I = 1 to Size do
        begin
            Temp6 := Temp6^.NextVector;
            Temp1 := Temp1^.NextVector;
            Temp4 := PoppedElement;
            For J = 1 to Size do
                begin
                    Temp4 := Temp4^.NextVector;
                    Temp2 := Temp4^.ThisVector;
                    Temp5 := Temp6^.ThisVector;
                    While Temp2 <> nil do
                        begin
                            Temp5^.Data := Temp5^.Data + TheMatrix[I, J]*
                                            Temp2^.Data;
                            Temp2 := Temp2^.Next;
                            Temp5 := Temp5^.Next;
                        end;
                end;
        end;

    NewBasis := AnotherHead;
end \{New Basis\}

Var

I, J, K : Integer;

begin

   CoefMatrix := ZeroMatrix (CoefMatrix, Size);

   For I := 1 to Size do
      For J := 1 to Size do
         For K := 1 to Size do

   end. {MultiplyMatrix}

Procedure GetNewBasis (HeadOfBasis : HeadPointer;

   Var AnotherHead : HeadPointer; Var NValue : Item;

   CoefMatrix : PtoToMatrix; Dim : Integer; Var Index : Integer)

Var

TempHead : HeadPointer;

begin

   Index := HeadOfBasis . NextVector . Position;

   TempHead := NewBasis (HeadOfBasis . NextVector, CoefMatrix, Dim - 1);

   HeadOfBasis . NextVector := TempHead . NextVector;

   GetProjection (AnotherHead, Dim, HeadOfBasis, NValue);

   end. {GetNewBasis}
Procedure Checking (Var RightBasis : boolean; Piconstant : real; Dim : Integer

HeadOfBasis : HeadPointer; AnotherHead : HeadPointer);

Var

Finish : boolean
Temp2 : ElementPointer;
Temp1 : HeadPointer;
Norm1, Dum : real;

begin

Finish := false;
Norm1 := 0;
Temp1 := HeadOfPointer^.NextVector;
Temp2 := Temp1^.ThisVector;
While (Temp2 <> nil) do
begin

Norm1 := Norm1 + (sqr(Temp2^.Data));
Temp2 := Temp2^.Next;
end;

Temp1 := AnotherHead;
While (Temp1^.NextVector <> nil) and (Not Finish) do
begin

Temp1^.NextVector;
Dum := 0;
Temp2 := Temp1^.ThisVector;

While Temp2 <> nil) do
begin

Dum := Dum + (sqr(Temp2^.Data));
Temp2 := Temp2^.Next;
end;

If (Dum < (Piconstant * Norm1)) then
begin

RightBasis := false;
end;
end.
\[ \text{Finish} = \text{true} : \]
\[ \text{Dispose} (\text{AnotherHead}) : \]
\[ \text{end} \]
\[ \text{else} \]
\[ \text{RightBasis} := \text{true} . \]
\[ \text{end} : \]
\[ \text{If} \ \text{RightBasis then} : \]
\[ \text{begin} \]
\[ \text{Dispose} (\text{AnotherHead}) : \]
\[ \text{Dispose} (\text{HeadOfBasis}) : \]
\[ \text{end} : \]
\[ \text{end, \{Checking\}} \]
Procedure GetMatrix (CoefMatrix, PtrToMatrix, Var TempMatrix, PtrToMatrix)

Var

I, J, Temp: Integer;

begin

New(TempMatrix);

For I = 1 to Size do

  For J = 1 to Size do

    If (I = 1) and (J = 1) then

      TempMatrix[I, J] := 1

    else

      If I = 1 then

        TempMatrix[I, J] := 0

      else

        If J = 1 then

          TempMatrix[I, J] := NV:Value[I - 1]

        else

          TempMatrix[I, J] := CoefMatrix[I - 1, J - 1];

    If Index <> 1 then

    begin

      For J = 1 to Size do

      begin

        Temp := TempMatrix[Index, J];

        TempMatrix[Index, J] := TempMatrix[I, J];

        TempMatrix[I, J] := Temp;

      end;

      For I := 1 to Size do

      begin

        Temp := TempMatrix[I, Index];

      end;

      Index := 2;

    end;

end;
TempMatrix \[I, \text{Index}\] = TempMatrix \[I, 1\] \\
TempMatrix \[I, 1\] = Temp

end:

end.

end. \{GetMatrix\}
Procedure SetUpMatrix [Var CoefMatrix, PtrlToMatrix, NValue, Item, Index, Size : Integer]

Var IsSave : ArrayBoolean, Var StackOfMatrix, ArrayOfMatrix;

Var IndexOfMatrix : Integer, RightMatrix : Boolean

Var TheMatrix, TempMatrix, PtrlToMatrix, I : Integer

begin

GetMatrix (CoefMatrix, TempMatrix, NValue, Index, Size);

If IsSave[Size] then

begin

PopMatrix (StackOfMatrix, IndexOfMatrix, TheMatrix);

MultiplyMatrix (CoefMatrix, TempMatrix, TheMatrix, Size);

IsSave[Size] := false;

end

else

CoefMatrix := TempMatrix;

If Not RightBasis then

begin

PushMatrix (StackMatrix, IndexOfMatrix, CoefMatrix);

IsSave[Size] := True;

end;

end; {SetUpMatrix}
Procedure Reposition(Var ABasis HeadPointer Total Integer)

Var

Temp1 Temp2 Temp3 TempHead HeadPointer
I Integer

begin

I = 0
New Temp1
With Temp1 do

begin

Position = 0
ThisVector := nil
NextVector := nil

end:

TempHead := Temp1

Repeat

I := I + 1
Temp3 := ABasis
Temp2 := ABasis | NextVector
While Temp2 | Position <> I do

begin

Temp3 := Temp2
Temp2 := Temp2 | NextVector

end:

Temp3 | NextVector := Temp2 | NextVector
Temp1 | NextVector := Temp2
Temp1 := Temp | NextVector
Temp1 | NextVector := nil

Until (I = Total)

ABasis := TempHead

end; {Reposition}
Procedure Reduce (HeadOfBasis, HeadPointer, Var CoefMatrix, PtrToMatrix, Dim, Integer).

Var

MinNorm : real;

begin

Repeat

If Dim = 1 then

begin

HeadOfBasis := AnotherHead;

If FirstRound then

begin

FindNorm (Dim, Norm, HeadOfBasis);

Determinant := Determinant * Norm[i];

FirstRound := false;

end;

New (CoefMatrix);

CoefMatrix[i, 1] := 1;

RightBasis := true;

end

else

begin

If Not IsRecursive then

HeadOfBasis := AnotherHead

else

Reposition (HeadOfBasis, Dim);

IsRecursive := false;

FindNorm (Dim, Norm, HeadOfBasis);

GetMinNormAndRNum (Dim, Norm, HeadOfBasis, MinNorm);

GetProjection (AnotherHead, Dim, HeadOfBasis, NValue);

If FirstRound then

Determinant := Determinant * MinNorm;

AccuCount := AccuCount + 1;

end

end;
\textbf{Reduce} \((HeadOfBasis, CoefMatrix, Dim - 1)\).

If \(\text{Count} \leq 72\) then

begin

If \(\text{Determinant} \neq 0\) then

begin

\text{IsRecursive} := \text{true} ;

\text{GetNewBasis} \((HeadOfBasis, AnotherHead, NVvalue, CoefMatrix, Dim, Index)\);

\text{Checking} \((RightBasis, PiConstant, Dim, HeadOfBasis, AnotherHead)\);

\text{SetUpMatrix} \((CoefMatrix, NVvalue, Index, Dim, IsSave, StackOfMatrix, IndexOfMatrix, RightBasis)\);

end

\text{else}

\text{Exit} \text{(Reduce)} ;

\text{end}

\text{else}

begin

\text{writeln} \(\ldots\text{Stack overflow}....\) ;

\text{writeln} \(\ldots\text{Re-enter your constant, must be greater than } 4/3\) ;

\text{Exit} \text{(Reduction)} ;

\text{end}

\text{end}

\text{end; \text{(Reduce)}}

\text{Until} \text{(RightBasis)} ;

\text{end; \{Reduce\}}
Main Program

begin

writeln ('Valid Dimension is in between 1 and ', ItemMax).

Rewrite (OutData, 'Printer').

AccuCount = 0.

Repeat

Page (OutData).

InitArray Boolean (IsSave).

GetBasis (AnotherHead, IsOkay, Dimension).

If IsOkay then

begin

writeln (OutData, ': NumElement = 2, 'Input Basis'.

'...Norm.' : NumElement = 8);

writeln (OutData);

FindNorm (Dimension, Norm, AnotherHead).

WriteMyResult (AnotherHead, Norm).

RightBasis := false;

IsRecursive := false;

FirstRound := true;

writeln ('..Enter Your Constant, must be greater than 4/3...');

readln (Aconstant);

writeln ('..Your constant is...', Aconstant : 7:2);

writeln ('OutData,'...Your constant is := Aconstant : 9:2');

writeln ('-----------------------------------------------');

If (Aconstant > 4/3) then

begin

Pconstant := 1/(4 * (Aconstant - 1));

Dim := Dimension;

IndezOfMatrix := 0;

Determinant := 1;

Count := 1;
OriginalBasis := Copy (AnotherHead, Dim),
Reduce (AnotherHead, CoefMatrix, Dim),
IsOkay := true;
If (Determinant = 0) then
begin
  writeln ('Your input basis is linearly dependent vectors'),
  IsOkay := false;
end;
If (IsOkay) then
begin
  OriginalBasis := NewBasis (OriginalBasis,
  CoefMatrix, Dimension);
  writeln (OutData);
  writeln (OutData, ',', NumElement * 2, ', Reduce Basis',
  ', ', NumNorm : NumElement * 8);
  writeln (OutData);
  FindNorm (Dimension, Norm, OriginalNorm);
  WriteMyResult (OriginalBasis, Norm);
  writeln (OutData);
  writeln (OutData, ', Determinant of Lattice :=',
  Determinant : 12 : 4);
  writeln (OutData);
  writeln (OutData, ', Total number of recursive :=',
  Count);
  writeln (OutData);
  writeln (OutData, '=====================================
  end;
end;
else
  writeln ('Error in your data'),
end;
writeln ('Type 'E' or 'e' to Stop or Type any character to continue')
readln (command);
Until ((Command = 'E') or (Command = 'e'));
Page (OutData);
Page (OutData);
end. {Reduction}
Implementation of The Reduction Process II

Program

Global Variable

Const  \textit{ItemMax} = 10.

Type \textit{ArrayCoeff} = Array[1..\text{ItemMax}] of \texttt{real}.

\textit{ArrayCoeff} = Array[1..\text{ItemMax}].

\textit{ElementPointer} = | \textit{ListElement}.

\textit{ListElement} =

record

Data : \texttt{real};

Next : \textit{ElementPointer};
end.

\textit{HeadPointer} = | \textit{HeadElement};

\textit{HeadElement} =

record

Position : \texttt{integer};

ThisVector : \textit{ElementPointer};

NextVector : \textit{HeadPointer};
end;

Var  \textit{OutData} : text;

\textit{IsOkay} : boolean;


\textit{HeadOfBasis}, \textit{Ahead} : \textit{HeadPointer};

\textit{Command} : \texttt{char};

\textit{Norm} : \textit{ArrayNorm};

\textit{Note : Procedure GetBasis present here, see from the reduction process II}.
Procedure WriteMyResult (HeadOfBasis, HeadPointer, Norm, ArrayNorm)

Var Temp1, HeadPointer, Temp2, ElementPointer, I : Integer;

begin

Temp1 := HeadOfBasis 'NextVector
I := 0;

while Temp1 <> nil do
begin

I := I + 1;
write(OutData, '1, ');
Temp2 := Temp1 'ThisVector
while Temp2 <> nil do
begin

write(OutData, Temp2 'Data:9, ');
Temp2 := Temp2 'Next;
end;

writeln(OutData, '1, ' .sqrt(Norm[I]):7, ');
Temp1 := Temp1 'NextVector;
end;

writeln(OutData, '12, ');
writeln(OutData);

end. (WriteMyResult)
Function NewHead(Dimension, NumElement, Integer) HeadPointer

Var Basis Temp1 HeadPointer
    Temp2 ElementPointer;
    I J Integer

begin
new(Basis).
with Basis ' do
begin
    Position = 0.
    ThisVector = nil.
    NextVector = nil.
end.
NewHead = Basis.
Temp1 = Basis.
for I = 1 to Dimension do
begin
new(Basis).
    with Basis ' do
    begin
        Position = I.
        ThisVector = nil.
        NextVector = nil.
    end;
Temp1 | NextVector := Basis;
Temp1 = Temp1 | NextVector;
new(Vector);
    with Vector ' do
    begin
        Data := 0;
        Next = nil;
    end;
end;
\texttt{Temp1 \mid ThisVector := Vector;}
\texttt{Temp2 := Temp1 \mid ThisVector;}
\texttt{for } J := 2 \text{ to } \text{NumElement do}
\begin{align*}
\text{begin} \\
\ & \text{new(Vector);} \\
\ & \texttt{\#th Vector \mid do} \\
\ & \text{begin} \\
\ & \quad \texttt{Data := 0;} \\
\ & \quad \texttt{Next := nil;} \\
\ & \text{end;} \\
\ & \texttt{Temp2 \mid NextVector := Vector;} \\
\ & \texttt{Temp2 := Temp2 \mid NextVector;} \\
\text{end;}
\end{align*}
\text{end: } \{ \text{New Head} \}
Function Copy (Head : HeadPointer, Dimension : Integer) : HeadPointer;

Var AHead, Temp1, Temp2 : HeadPointer;
    Temp3, Temp4 : ElementPointer;

begin
    AHead := NewHead(Dimension, NumElement);
    Temp1 := Head;
    Temp2 := AHead.
    while Temp1 <> nil do
        begin
            Temp1 := Temp1.NextVector;
            Temp2 := Temp2.NextVector;
            Temp3 := Temp3.NextVector;
            Temp4 := Temp4.NextVector;
            while Temp3 <> nil do
                begin
                    Temp4.Data := Temp3.Data;
                    Temp3 := Temp3.NextVector;
                    Temp4 := Temp4.NextVector;
                end;
        end;
end; {Copy}
Function FindNorm (FirstVector, SecondVector, HeadPointer) real.

Var Temp1, Temp2, ElementPointer, Dum real.

begin
  Temp1 := FirstVector ⊕ ThisVector;
  Temp2 := SecondVector ⊕ ThisVector;
  Dum := 0;
  while Temp2 <> nil do
  begin
    Dum := Dum + (Temp1 ⊕ Data ⊕ Temp2 ⊕ Data);
    Temp2 := Temp2 ⊕ Next;
    Temp1 := Temp1 ⊕ Next;
  end;

  FindNorm := Dum;

end;
Procedure Subtract (FirstVector, SecondVector, HeadPointer, A_Value, real)
Var Templ, Temp2, ElementPointer.
begin
    Templ = FirstVector \ ThisVector;
    Temp2 = SecondVector \ ThisVector;
    while Templ <> nil do
        begin
            Templ \ Data = Templ \ Data - (A_Value * Temp2 \ Data);
            Temp2 = Temp2 \ Next.
            Templ = Templ \ Next.
        end.
    end. (Subtract)

Function Search (AnyBasis, HeadPointer, WhichPosition, Integer) HeadPointer.
Var Templ, HeadPointer.
begin
    Templ = AnyBasis \ NextVector;
    If WhichPosition > 0 then
    begin
        while (Templ <> nil) and (Templ \ Position <> WhichPosition) do
            Templ = Templ \ NextVector;
        Search = Templ.
    end
    else
    Search = AnyBasis.
end. (Search)
Procedure GetProjection (HeadOfBasis : Head Pointer, Var Norm : Array Norm,
Var Coefficient : Array Coef):
Var Temp1, Temp2, AnotherHead, Temp3 : Head Pointer;
I, J : integer;
Dummy, Dummy1 : real;
begin
  AnotherHead := Copy (HeadOfBasis, Dimension);
  Temp1 := AnotherHead \ NextVector;
  Temp3 := HeadOfBasis \ NextVector;
  for I := 1 to Dimension do
    begin
      Temp2 := AnotherHead \ NextVector;
      for J := 1 to I - 1 do
        begin
          Dummy := FindNorm (Temp3, Temp2);
          Coefficient[I, J] := Dummy / Norm[J];
          Dummy1 := Coefficient[I, J] ;
          Subtract (Temp1, Temp2, Dummy1);
          Temp2 := Temp2 \ .NextVector;
        end;
      Norm[I] := FindNorm (Temp1, Temp1);
      Temp1 := Temp1 \ .NextVector;
      Temp3 := Temp3 \ .NextVector;
    end;
end; {GetProjection}
Procedure Reduce_A_Basis (HeadOfBasis HeadPointer; K_Value, L_Value Integer.
    Var Coefficient Array Coef).

Var Temp1, Temp2 HeadPointer,
    Round_Value, J Integer.

begin
    Temp1 := Search (HeadOfBasis, K_Value);
    Temp2 := Search (HeadOfBasis, L_Value);
    Round_Value := Round (Coefficient[K_Value, L_Value]).
    Subtract (Temp1, Temp2, Round_Value);
    for J = 1 to L_Value - 1 do
        Coefficient[K_Value, J] := Coefficient[K_Value, J] -
        (Round_Value * Coefficient[L_Value, J]);
        Coefficient[K_Value, L_Value] := Coefficient[K_Value, L_Value]
        - Round_Value;

end; {Reduce_A_Basis}
Procedure Modify_A_Basis (HeadOfBasis : HeadPointer; Var Norm : ArrayNorm;
Var Coefficient : ArrayCoef, K_Value : Integer);

Var Temp, Temp1, Temp2 : HeadPointer;
Dummy, Dummy1, Dummy2 : real,
J : integer;
Save : ArrayCoef;

begin

Dummy := Coefficient[K_Value, K_Value - 1];
Dummy1 := Norm[K_Value] + (sqrt(Dummy) * Norm[K_Value - 1]);
Coefficient[K_Value, K_Value - 1] := (Dummy * Norm[K_Value - 1]) / Dummy1;
Norm[K_Value] := (Norm[K_Value - 1] * Norm[K_Value]) / Dummy1;
Temp := Search(HeadOfBasis, k_Value - 2);
Temp1 := Search(HeadOfBasis, k_Value - 1);
Temp2 := Search(HeadOfBasis, k_Value);
Temp1 | .Position := K_Value;
Temp2 | .Position := K_Value - 1;
Temp1 | .NextVector := Temp2;
Temp1 | .NextVector := Temp2 | .NextVector;
Temp2 | .NextVector := Temp1;
for J := 1 to K_Value - 2 do

begin

Dummy2 := Coefficient[K_Value, J];
Coefficient[K_Value, J] := Coefficient[K_Value - 1, J];
Coefficient[K_Value - 1, J] := Dummy2;
end;

for J := K_Value + 1 to Dimension do

begin

Save[J, K_Value - 1] := (Coefficient[K_Value, K_Value - 1] * Coefficient[J, K_Value - 1]) + (Coefficient[J, K_Value] * (1 - Dummy * Coefficient[K_Value, K_Value - 1]));
Save[J, K_Value] := Coefficient[J, K_Value - 1] - (Dummy * Coefficient[J, K_Value])
end;

for J := K_Value + 1 to Dimension do
begin
\hspace{1em}Coefficient[J,K,Value - 1] := Save[J,K,Value - 1];
\hspace{1em}Coefficient[J,K,Value] := Save[J,K,Value];
end;
end. \{Modify_{A,Basis}\}
Procedure Reduce (HeadOfBasis : HeadPointer; Var Norm : ArrayNorm).
Var
  Finished, Okay : boolean;
  K_Value, L_Value, L : Integer;
  Coefficient : ArrayCoeff;
  Check_Value : real;

begin
  GetProjection (HeadOfBasis, Norm, Coefficient);
  K_Value := 2;
  Count1 := 0;
  Count2 := 0;
  Finished := false;
  while not Finished do
    begin
      L_Value := K_Value - 1;
      if Abs(Coefficient[K_Value, L_Value]) > 0.5 then
        Reduce_A_Basis (HeadOfBasis, K_Value, L_Value, Coefficient);
        Check_Value := (0.75 - Sqr(Coefficient[K_Value, K_Value - 1]))
          * Norm[K_Value - 1];
        if Norm[K_Value] < Check_Value then
          begin
            Modify_A_Basis (HeadOfBasis, Norm, Coefficient, K_Value);
            if K_Value > 2 then
              begin
                Count1 := Count1 + 1;
                K_Value := K_Value - 1;
              end;
            Okay := false;
          end
          else
            Okay := true;
        if Okay then
          begin
            if K_Value > 2 then
              for L := K_Value - 2 downto 1 do
                if Abs(Coefficient[K_Value, L]) > 0.5 then
Reduce A Basis (HeadOfBasis, K_Value, L_Coefficient);

if K_Value = Dimension then
  Finished = true
else
  begin:
    K_Value := K_Value + 1;
    Count2 := Count2 + 1;
    Finished := false;
  end;
end:
else
  Finished := false;
end: {Reduce}
Main Program

begin

writeln('Valid dimension is in between 1 and 8');
rewrite(OutData, 'Printer');

repeat

page(OutData);

GetBasis (HeadOfBasis, IsOkay, Dimension);

if IsOkay then

begin

writeln(OutData, 'Dimension = 2; Input basis');

writeln(OutData);

AHead := HeadOfBasis \ .NextVector;

for I := 1 to Dimension do

begin

Norm[I] := FindNorm(AHead, AHead);

AHead := AHead \ .NextVector;

end;

WriteMyResult (HeadOfBasis, Norm);

Reduce (HeadOfBasis, Norm);

writeln(OutData, 'Dimension = Reduced Basis');

writeln(OutData);

AHead := HeadOfBasis \ .NextVector;

for I := 1 to Dimension do

begin

Norm[I] := FindNorm(AHead, AHead);

AHead := AHead \ .NextVector;

end;

WriteMyResult (HeadOfBasis, Norm);

writeln(OutData);
writeln(OutData,'Count1: ',Count1,' Count2: ',Count2);
end
else
writeln('Error in input basis');
writeln('Type "E" to stop or type any character to continue');
readln(Command);
until ((Command = 'E') or (Command = 'e')).
page (OutData).
page (OutData).
end. {Reduction}
Chapter 4

Some Interesting Algorithms

Besides the algorithm in Chapter 2, there are 2 more algorithms which can, in polynomial time, solve the ILP problems in special cases. We will present these two algorithms in Section 4.2 and Section 4.3. In fact, the algorithm by KANNAN [9] is similar to the one by HIRSCHBERG and WONG [8], since KANNAN's algorithm breaks the ILP problem into at most \( n \)-knapsack problems. Both algorithms solve the ILP problems with only 2 variables. Unlike the HIRSCHBERG and WONG's algorithm, KANNAN's algorithm is a recursive one. Both algorithms can solve the ILP problems in their special cases within time bounded by a polynomial function of the length of input.

In Section 4.1, we will show that the algorithm in Chapter 2 can also solve the ILP problems with a fixed value of \( m \) in a polynomial time, where \( m \) is the number of constraint equations.

4.1 A polynomial time algorithm for the integer linear programming problem with a fixed number of constraints.

This algorithm was noted by P. Van Emde Boas as an immediate consequence of the main result from the algorithm in Chapter 2. The algorithm is as follows.

Let the ILP problem be as the one in Chapter 2, we have to decide whether there exists a vector \( z \in \mathbb{Z}^n \) for which \( Ax \leq b \), where \( A \) is \( (m \times n) \)-matrix, \( b \) is a \( m \)-vector. By repeated application
of the Euclidean algorithm (as shown in Chapter 2), we can find an \((n \times n)\) matrix \(U\) with integral coefficients and determinant \(\pm 1\) such that the matrix

\[
AU = \{a_{i,j}\}_{1 \leq i \leq m, 1 \leq j \leq n}
\]

is a lower triangle matrix with

\[
a_{i,j} = 0 \quad \text{for} \quad j > i.
\]  

(4.1)

Let \(y = U^{-1}x\); we can see that the existence of \(x \in \mathbb{Z}^n\) with \(Ax \leq b\) is equivalent to the existence of \(y \in \mathbb{Z}^n\) with

\[
(AY)y \leq b \quad \text{(from} \quad x = Uy).\]

If \(n > m\), then the coordinates \(y_{m+1}, \ldots, y_n\) of \(y\) do not occur in these inequalities, since (4.1) implies that \(a'_{i,j} = 0\) for \(j > m\). Hence, the original problem can be reduced to a problem with only \(\min\{n, m\}\) variables. In both cases, we can use the algorithm in Chapter 2 to solve them. It follows from the main result of Chapter 2, we can conclude that any ILP problem with a fixed value of \(m\) is a polynomially solvable problem.

4.2 A polynomial time algorithm for the knapsack problem with two variables

The knapsack problem considered here is the following.

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{n} c_i x_i \\
\text{subject to} & \quad \sum_{i=1}^{n} a_i x_i \leq b \\
\text{and} & \quad x, \text{ non-negative integers}.
\end{align*}
\]
where \( a, c, b \) are positive numbers. In general, we consider the knapsack problems as the ILP problems with any 1 constraint.

In [8], HIRSCHBERG and WONG give an algorithm to solve the knapsack problems with only 2 variables. This algorithm runs in polynomial time of length of the input. Thus, our problem is

**Problem 1**

\[
\text{maximize } c_1 x_1 + c_2 x_2
\]

subject to \( a_1 x_1 + a_2 x_2 \leq b \)

and \( x_1, x_2 \) non-negative integers.

*Note that* the naive approach (test \( x_1 = i, x_2 = [(b - a_1 i)/a_2] \) as \( i \) takes on integer value from 0 to \( [b/a_1] \)) take time proportional to \( b/a_1 \) which is exponential in the length of the input.

This algorithm originates from a well-known continued fraction method. We first transform **Problem 1** into **Problem 2** such that

**Problem 2**

\[
\text{maximize } x_1 + k_1 x_2
\]

subject to \( x_1 + k_2 x_2 \leq k_3 \)

\( x_1, x_2 \) non-negative integers

and \( k_1 = c_2 / c_1, k_2 = a_2 / a_1, k_3 = b / a_1 \)

(without loss of generality, we can assume that \( k_3 > k_1 \)).

The algorithm generates an integer sequence \( b_i \) \((0 \leq i \leq k)\), such that the last \( b \) value doesn't exceed the value of \( k_3 / k_2 \). We consider the value of \( b_i \) such that
4.3 A polynomial algorithm for the two-variable integer programming problem

We consider the following integer programming problem:

Problem 1

\[ \text{maximize } c_1 x_1 + c_2 x_2 \]

subject to \( a_{1i} x_1 + a_{2i} x_2 \leq b_i \), \( i = 1, 2, \ldots, n \)

and \( x_1, x_2 \) non-negative integers

where \( a_{1i}, c_1 \) and \( b_i \) are assumed to be non-negative integers. We can obtain the solution of Problem 1 from the solutions to at most \( n \) problems, each of which is of the form:

Problem 2

\[ \text{maximize } c_1 x_1 + c_2 x_2 \]

subject to \( a_{1i} x_1 + a_{2i} x_2 \leq b \)

\[ L \leq x_1 \leq U \]

and \( x_1, x_2 \) non-negative integers
where all the constants involved are positive integers, \( U \leq \lfloor b/a_1 \rfloor \). If any one of the following three conditions is met, then Problem 2 is obviously trivially solved.

a) \( c_1 \) (or \( c_2 \)) \( \leq 0 \) : \( x_1 \) (or \( x_2 \)) can be set to its lowest permissible value and the resulting one-variable problem solved.

b) \( a_1 \) or \( a_2 = 0 \), or \( b \leq 0 \)

c) \( L = U \)

If none of these three conditions is satisfied, the algorithm by KANAN [9] successively reduces Problem 2 to other problems of the same form until one of the conditions a), b) or c) is met. We can assume, without loss of generality, that \( a_1 U \leq b \). If not replace \( U \) by \( \lfloor b/a_1 \rfloor \). This algorithm can be performed recursively until the new problem with one of those three conditions can be constructed.

Thus, to solve Problem 1, we only need to find the maximum of \( c_1 x_1 + c_2 x_2 \) over the integer points in each set of the at most \( n \) new problems, and then take the maximum among all these maxima.

KANAN shows that the problem can be solved within a time bounded by a polynomial function of the length of the input.
REFERENCES


