INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor MI 48106-1346 USA
313/761-4700  800/521-0600
SERIES SOLUTIONS OF THE HÉNON-HEILES SYSTEM

by

SUSAN PAY, B.A.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics and Statistics
Carleton University
Ottawa, Canada
May, 1997

© Copyright 1997
Susan Pay
The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author’s permission.

L’auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L’auteur conserve la propriété du droit d’auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.
The undersigned recommend to
the Faculty of Graduate Studies and Research
acceptance of the thesis

SERIES SOLUTIONS OF THE HÉNON-HEILES SYSTEM

submitted by

SUSAN PAY, B.A.

in partial fulfillment of the requirements for
the degree of Master of Science
Information System Science

[Signature]
Thesis Supervisor

[Signature]
Chair, Department of Mathematics and Statistics

Carleton University
Abstract

This thesis deals with series solutions of the Hénon-Heiles system. For certain values of its parameters, the general solution of the Hénon-Heiles system can be expressed as a series about arbitrary poles. In all other parameter regimes, the general solution contains movable branch points in which case the expansion is called a psi-series. In this thesis we shall discuss the convergence of the psi-series solutions of the Hénon-Heiles system.

The psi-series of the Hénon-Heiles system have several different forms depending upon the values of the parameters. We prove that all these psi-series are convergent, thus establishing that the formal solutions are actual solutions.
Acknowledgments

I wish to express my deep gratitude to Dr. Sam Melkonian for his continued support and for his remarkable suggestions. I would also like to thank Feng and Ning for their warm support.
Contents

Abstract

Acknowledgments

INTRODUCTION............................................................................................................. 1

1. SERIES SOLUTIONS OF HÉNON-HEILES SYSTEM ........................................... 1-1

1.1. The H-H system ................................................................................................. 1-1
1.2. The basic definitions .......................................................................................... 1-1
1.3. The integrable H-H system ............................................................................... 1-2
   1.3.1. The ARS algorithm ..................................................................................... 1-2
   1.3.2. The weak Painlevé test .............................................................................. 1-7
1.4. The non-integrable H-H system ........................................................................ 1-9

2. THE PSI-SERIES SOLUTIONS ............................................................................ 2-1

2.1. The H-H system with a positive incomplete resonance .................................. 2-1
   2.1.2. The convergence proof .............................................................................. 2-2
2.2. Complex or irrational conjugate resonance ..................................................... 2-11

3. CONVERGENCE PROOF FOR PSI-SERIES ...................................................... 3-1

3.1. Case (I): Convergence proof of the H-H system ............................................ 3-1
3.2. Case (II): Convergence proof of the H-H system .......................................... 3-10

4. COMPLEX TIME .................................................................................................... 4-1

CONCLUSION ............................................................................................................ 5-1

Appendix ..................................................................................................................... 6-1

Bibliography .............................................................................................................. 7-1
Introduction

The Hénon-Heiles system is an important dynamical system [26], [8], [13]. In recent years, much research has been focused on its integrable cases [8], [9], [20]. As for the non-integrable cases, the singularity analysis by Chang *et al* [11] showed that the complex singularities of the H-H system cluster recursively in the complex domain, forming self-similar natural boundaries.

For those parameter values for which the H-H system is integrable, the general solution is expressible as a Laurent series [13]. For all other parameter values, the system is non-integrable and the general solution takes the form of a psi-series, which is an expansion about an arbitrary branch point [11].

In order to show that the psi-series solutions are actual solutions of the H-H system, it is necessary to prove their convergence. Chang *et al* [11] have stated that the psi-series solutions of the H-H system are convergent, but have not given a proof. Hemmi and Melkonian [17] have described a method for proving the convergence of second- and third-order ordinary differential equations (ODEs). Melkonian and Zypchen [22] have shown the convergence of the psi-series solutions of the Lorenz system and the Duffing equation. All of those convergence proofs have similar approaches. In this thesis, we give a proof of the convergence of the psi-series solutions of the H-H system based upon the method discussed in [17].

The organization of the thesis is as follows: In Chapter 1, we briefly discuss integrable cases of the H-H system by applying the ARS algorithm. In Chapter 2, we discuss the different forms of the psi-series solutions of the H-H system. In Chapters 3, we give the convergence proofs for real time. Complex time is dealt with in Chapter 4.
Chapter 1

Series Solutions of the Hénon-Heiles System

The general solution of the Hénon-Heiles system admits meromorphic Laurent series representations about movable poles only for certain values of its parameters. In all other parameter regimes, the general solution contains movable logarithmic branch points. Series expansions about such points are termed psi-series and constitute formal general solutions.

When the general solution of the Hénon-Heiles system is a meromorphic Laurent series, the H-H system is integrable. A valuable test for the integrability of a system is the Painlevé test (the ARS test). By applying the ARS test to the H-H system, we shall see that there are only a few parameter values for which the H-H system is integrable; for most parameter values, the system is non-integrable.

1.1 The H-H system

In this thesis, we shall be concerned with the cubic Hénon-Heiles system, which is a Hamiltonian system, with the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + Ax^2 + By^2) + Dx^2y - \frac{1}{3}Cy^3$$

(1.1)

and the equations of motion

$$x'' = -Ax - 2Dxy$$

$$y'' = -By - Dx^2 + Cy^2,$$

(1.2)

where $A$, $B$, $C$, $D$ are arbitrary parameters.

In order to determine the integrability properties of the above Hénon-Heiles system, we need to give some basic definitions.

1.2 Basic Definitions:

A system is generally said to be integrable if it has the Painlevé property, i.e., its general solution has no movable singularities other than poles. (A more precise definition of integrability continues to be elusive, although several “types” of integrability have been discussed in the literature ([8], [13], [16], [22], [31]).

1-1
**Singualrities** (points at which a solution is not analytic) can be poles, branch points (algebraic or transcendental (e.g., logarithmic or irrational)), or essential. Each one of these can be fixed or movable.

The **location** of a fixed singularity is determined by the coefficients of the differential equation. In contrast, the location of a movable singularity is dependent upon the initial conditions. All singularities of linear differential equations are fixed. However, nonlinear equations may possess singularities of either type.

The solution of a differential equation with fixed singularities can be analytically continued to provide a single-valued solution. On the other hand, if a differential equation possesses movable singularities which are not poles, analytic continuation of its solution may be impossible.

### 1.3 The integrable H-H system

The ability to identify integrable systems is very important since, in a certain sense, such systems are “exactly soluble” and their solutions can be represented analytically. These solutions can be used as “building blocks” about which more complicated cases can be studied. A valuable test for the integrability of a system is the Painlevé test (the ARS test). The integrable case of H-H system was discussed in [9, 10, 28, 29, 13].

#### 1.3.1 The Painlevé Test (The ARS Algorithm)

The ARS algorithm was developed by Ablowitz, Ramani and Sugar. It is a method for determining the nature of the movable singularities of nonlinear ordinary differential equations (ODEs) without knowing the exact general solution. It was originally developed to determine whether a non-linear ODE (or a system of ODEs) admits movable branch points (either algebraic or logarithmic). It is important to note that the movable essential singularities cannot be detected by this algorithm. (We shall see this in the H-H system later on). It only provides necessary conditions for a given non-linear ODE to possess the Painlevé property.

The ARS algorithm has been discussed in detail in [17]. Here, we apply this algorithm to the H-H system.
The leading-order analysis:

Let $x = a(t - t_0)^\alpha$ and $y = b(t - t_0)^\beta$,

where $\alpha$ and $\beta$ are assumed to be negative. Substitute into the equation of motion (1.1) and balance the leading-order terms to obtain:

$$a\alpha (\alpha - 1)(t - t_0)^{\alpha-2} = -2Dab(t - t_0)^{\alpha+\beta},$$

$$b\beta (\beta - 1)(t - t_0)^{\beta-2} = -Da^2(t - t_0)^{2\alpha} - Cb^2(t - t_0)^{2\beta}.$$  

The possible leading-order balances are as follows [25]:

**Case (I):** $\alpha = -2$, $a = \pm \frac{3}{D} \sqrt{2 + \frac{1}{\lambda}}$, where $\lambda = \frac{D}{C}$,  

$$\beta = -2, \quad b = -\frac{3}{D}.$$  

If one assumes that $\alpha$ is less than $\beta$

**Case (II):** $\alpha = \frac{1}{2} \pm \sqrt{1 - 48\lambda}$, $\alpha$ is arbitrary, where $\lambda = D \frac{2}{C}$,

$$\beta = -2, \quad b = \frac{6}{C}.$$  

Case (II) can occur only for $\lambda > -\frac{1}{2}$. The above forms of $\alpha$, $\beta$ are obtained by equating the leading-order exponents in (1.12) and (1.13). In case (II), $\alpha$ has two roots. However, since the most singular behaviour that can be supported by the equations motion is $(t - t_0)^2$, both roots can exist only for $\lambda > -\frac{1}{2}$ [25].

The resonances:

For case (I), we set

$$x = \pm \frac{3}{D} \sqrt{2 + \frac{1}{\lambda}} \tau^2 + pt^{-2}$$

$$y = \frac{3}{D} \tau^2 + qt^{-2}$$

where $\tau = t - t_0$, and $p$ and $q$ are arbitrary parameters.
The linear equations for $p$ and $q$ resulting from the dominant terms in equation (1.11) are:

\[
\begin{bmatrix}
(r - 2)(r - 3) + 6\sqrt{2 + \frac{1}{\lambda}} \\
\pm 3
\end{bmatrix} p
\begin{bmatrix}
\pm 6\sqrt{2 + \frac{1}{\lambda}} \\
(r - 2)(r - 3) - \frac{6}{\lambda}
\end{bmatrix} q
= \begin{bmatrix}
0 \\
0
\end{bmatrix}, \tag{1.7}
\]

from which we find that the resonances occur at:

\[ r = -1, 6, \frac{5}{2} \pm \frac{1}{2} \sqrt{1 - 24(\frac{1}{\lambda} + 1)} . \]

For case (II), we set

\[ x = \alpha \tau^2 + \tau^{r - 2}, \]
\[ y = \frac{6}{C} \tau^2 + \alpha \tau^{r - 2}, \]

where $\alpha = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 48\lambda}$, $\alpha$ is arbitrary, and the dominant terms in (1.1) give:

\[
\begin{bmatrix}
[r + (\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 48\lambda})] \\
0
\end{bmatrix} p
\begin{bmatrix}
2\alpha \\
0 \quad (r - 2)(r - 3) - 12
\end{bmatrix} q
= \begin{bmatrix}
0 \\
0
\end{bmatrix} . \tag{1.8}
\]

In this case, the resonances are found to be

\[ r = -1, 0, 6, \pm \sqrt{1 - 48\lambda} . \]

Since the system (1.1) is a fourth-order equation, there must be four arbitrary constants in the local expansion of the general solution. One of these is the singularity location $t_0$ which corresponds to the resonance $r = -1$, and the another arises at $r = 6$. The remaining two depend upon the value $\lambda$.

In case (I), in order for the last two resonances to be nonnegative (we shall not consider negative resonance in this thesis), we must have $\lambda > 0$ or $\lambda < \frac{1}{2}$.
Furthermore, in order to have only integer resonances, \( \lambda \) must be restricted to one of the following values:

(a) \( \lambda = -1, \ r = -1, 2, 3, 6 \)

(b) \( \lambda = -\frac{1}{2}, \ r = -1, 0, 5, 6 \)

(c) \( \lambda = -\frac{1}{6}, \ r = -1, -3, 6, 8 \)

The resonance at zero corresponds to the arbitrariness of the leading-order coefficient. If one combines the case (I) and (II), it can be seen that in order for the H-H system to have the Painlevé property, one must have integer leading orders and resonances. This can only occur at \( \lambda = -1, \ \lambda = -\frac{1}{2}, \) or \( \lambda = -\frac{1}{6}. \)

Note:
1. At \( \lambda = 1 \) and \( \lambda = -\frac{1}{2}, \) the power of \( t \) at which the resonance occurs in case (I) is identical to the second possible leading-order for case (II). We call them the canonical resonance.
2. At \( \lambda = -\frac{1}{2}, \) the case (I) resonance at \( r = 0 \) corresponds to the vanishing of the leading-order coefficient \( a = \pm \frac{3}{2} \sqrt{2 + \frac{1}{\lambda}}. \) Since the original ansatz was \( x(t) = a(t - t_0)^2 \) to leading order, this is a contradiction. It turns out that the leading order is not -2, but involves logarithmic terms. The solution is a psi-series.

The last step in the ARS algorithm is to check the compatibility conditions to make sure that the expansion is self-consistent.

Calculation of the Arbitrary Constants:

By the foregoing discussion, the only \( \lambda \)-values for which the H-H system has the Painlevé property are \( \lambda = -1 \) and \( \lambda = -\frac{1}{6}. \)

For \( \lambda = -1, \) the general solution takes the form

\[
x(t) = \sum_{i=0}^{\infty} a_i t^{-i}.
\]  \hspace{1cm} (1.9)

\[
y(t) = \sum_{i=0}^{\infty} b_i t^{-i}.
\]  \hspace{1cm} (1.10)
The coefficient recursion relations are (setting $C = -D$)

\[
\begin{bmatrix}
(i-2)(i-3) - 6 & \pm 6 \\
\pm 6 & (i-2)(i-3) - 6
\end{bmatrix}
\begin{bmatrix}
a_i \\
b_i
\end{bmatrix} =
\]

\[
\begin{bmatrix}
-Aa_i - z - 2D \sum_{l=1}^{i-1} a_l b_l \\
-Bb_i - z - D \sum_{l=1}^{i-1} a_l b_l + b b,
\end{bmatrix}
\]

\hspace{1cm} (1.11)

Since $a_0 = \pm \frac{3}{D}$ and $b_0 = \frac{3}{D}$, we find

\[
\begin{bmatrix}
6 & \pm 6 \\
\pm 6 & -6
\end{bmatrix}
\begin{bmatrix}
a_i \\
b_i
\end{bmatrix} = \begin{bmatrix}
\mp 3A / D \\
-3B / D
\end{bmatrix}.
\]

\hspace{1cm} (1.12)

When $i = 2$, the determinant of the matrix on the left-hand side vanishes, corresponding to the resonance $i = 2$. The null vector of this matrix is $[1 \quad \pm 1]$. In order for $a_2$ or $b_2$ to be arbitrary, we must have

\[
\begin{bmatrix}
1 & \pm 1
\end{bmatrix}
\begin{bmatrix}
\mp 3A / D \\
-3B / D
\end{bmatrix} = 0.
\]

Thus, the compatibility condition is satisfied if and only if $A = B$.

At $i = 3$, we have

\[
\begin{bmatrix}
6 & \pm 6 \\
\pm 6 & -6
\end{bmatrix}
\begin{bmatrix}
a_3 \\
b_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

\hspace{1cm} (1.13)

so $a_3$ or $b_3$ is arbitrary. (For example, $a_3 + b_3 = 0 \Rightarrow b_3 = \delta, a_3 = -\delta$, with $\delta$ arbitrary.)

For $\lambda = -1$, the H-H system has the Painlevé property provided that $A = B$. If $A \neq B$, we have an incompatible positive resonance, and the solution in this case involves logarithmic psi-series, which will be discussed in Chapter 3.

Similarly, for $\lambda = -1/6$, the H-H system has the Painlevé property for all values of $A$ and $B$, so the system is integrable.
From the above, the H-H system is integrable for $\lambda = -1$ ($A=B$) and $\lambda = -1/6$. On the other hand, if the H-H system fails the Painlevé test, can it still be integrable?

If the H-H system fails the Painlevé test because of finite branching at the arbitrary singularity (i.e., both the leading order and the positive resonances are at worst rational), then one can make the change of variable $z = \tau^{[d,p\cdot dr]}$ (where $\tau = t - t_0$), where $d$ is the least common multiple of the denominator of the leading order $d_p$ and resonance $d_r$, and then apply the Painlevé test in the new variable $z$. This is called the weak Painlevé test (see e.g. [17]).

1.3.2 The Weak Painlevé Test

The weak Painlevé test applies to systems which fail the Painlevé test but for which the leading order and/or resonances are rational.

For our H-H system, this means that

for case (I), we consider values of $\lambda$ for which the resonances are positive and rational. Thus, we require that

$$-\frac{24}{23} < \lambda < -\frac{1}{2}$$

and

$$\sqrt{1 - 24\left(\frac{1}{\lambda} + 1\right)} = \frac{n}{m}$$

where $m$ and $n$ are integers.

As for case (II), in order to have a positive resonance, $\lambda$ must satisfy:

$$-\frac{1}{2} < \lambda < \frac{1}{48}$$

Now in order to satisfy the second condition:

Let $\lambda = \frac{1}{48} \left(1 - \left(\frac{m}{n}\right)^2\right)$, $1 < \frac{m}{n} < 5$,

which results in the leading order

$$\alpha = \frac{1}{2} \left(1 - \frac{m}{n}\right), \quad \beta = -2$$

The resonances are:

$$r = -1, 0, \frac{m}{n}, 6.$$
Now the expansions about the movable singularity take the form of

$$x(t) = \tau^{-\gamma} \sum_{j=0}^{\infty} a_j \tau^j,$$

$$y(t) = \tau^{-\gamma} \sum_{j=0}^{\infty} b_j \tau^j. \tag{1.14}$$

In order for these expansions to be valid, the compatibility conditions at the resonances $j = m, j = 6n$ must be satisfied (since $\gamma = 0$ is satisfied already). It turns out that these two compatibility conditions are satisfied only for $m/n = 2$, corresponding to $\lambda = -\frac{1}{16}$ provided that $B = 16A$. In this case the local expansions take the form [25]

$$x(t) = \tau^{-1/2} \sum_{j=0}^{\infty} a_j \tau^j,$$

$$y(t) = \tau^{-1} \sum_{j=0}^{\infty} b_j \tau^j. \tag{1.16}$$

where $a_j$ and $b_j$ are given in the following table (where $\mu$, $\theta$ and $\psi$ are arbitrary) [26]. In terms of the variable $\tau^{-1/2}$ the above expansions are Painlevé. The free coefficients are $a_0, a_2$ and $b_6$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$a_j$</th>
<th>$b_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mu$</td>
<td>$\frac{3}{8}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$\theta$</td>
<td>-$\frac{A}{2}$</td>
</tr>
<tr>
<td>3</td>
<td>$-\frac{\mu^3}{18}$</td>
<td>$\frac{\mu^2}{12}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{5}{4}A^2$</td>
<td>$-\frac{4}{5}A^2$</td>
</tr>
<tr>
<td>5</td>
<td>$-\frac{\mu^2\theta}{18}$</td>
<td>$\mu\theta/3$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1}{24}(\mu^3/108-2\mu\psi)$</td>
<td>$\psi$</td>
</tr>
</tbody>
</table>

There may be some other parameter values for which the H-H system has the weak Painlevé property. For further studies of the H-H system with the weak Painlevé property, see [15].
In summary, the H-H system is integrable if

\[ \lambda = -1, \quad \lambda = -\frac{1}{6}, \quad (A = B), \quad \lambda = -\frac{1}{16}, \quad (A = 16B) \]

The results were proven by Ito in [32] and were confirmed numerically in [9].

1.4 The non-integrable H-H system

As we have seen above, for most \( \lambda \) values, the H-H system is non-integrable.

In case (I),

if \( \lambda < -\frac{24}{23} \), the H-H system has complex resonances, and therefore fails the Painlevé test;

if \( -\frac{1}{2} > \lambda > -\frac{24}{23} \), the H-H system has real and positive resonances, and the system might be integrable;

if \( -\frac{1}{2} < \lambda < 0 \), the H-H system has a negative resonance;

if \( \lambda > 0 \), the H-H system has complex resonances.

As for case (II),

if \( \lambda > \frac{1}{48} \), both the leading orders and the resonances are complex;

if \( -\frac{1}{2} < \lambda < \frac{1}{48} \), the negative branch is real and can define a four-parameter solution;

if \( \lambda = 0 \), the singularity in \( x \) disappears, and the equation of motion is integrable.
Chapter 2

The Psi-Series Solutions

As we have seen in Chapter 1, for most $\lambda$ values, the H-H system fails the Painlevé test and requires a psi-series expansion. There are several types of psi-series. Here, we mainly discuss two cases. The first case involves an incompatible positive integer resonance and the second has complex or irrational conjugate resonances.

2.1 The H-H System With an Incompatible Positive-Integer Resonance

The psi-series solutions in this case take the form

$$x(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k-p} (\tau^k \log \tau)^l,$$  \hspace{1cm} (2.1)

$$y(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} \tau^{k-p} (\tau^k \log \tau)^l,$$  \hspace{1cm} (2.2)

where $p$ is the leading order and $k_0$ is the smallest positive-integer resonance.

For example, for $\lambda = -1$, $A \neq B$, the psi-series solution is

$$x(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k-2} (\tau^2 \log \tau)^l,$$  \hspace{1cm} (2.3)

$$y(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} \tau^{k-2} (\tau^2 \log \tau)^l,$$  \hspace{1cm} (2.4)

In order to show this is the psi-series constitute solutions of the H-H system, first we must check that it contains a sufficient number of free (arbitrary) coefficients. According to [9] and [25], the free coefficients in this case are $a_{20}$, $a_{30}$, $a_{60}$ and $l_0$. Next, we must show that the psi-series (2.3) and (2.4) are convergent.
2.1.1 The Convergence Proof

In order to show that the above psi-series (2.3) and (2.4) are convergent, we break the proof into several steps; first, the multiple series is resummed into a single series with function coefficients, then it is shown that the coefficient functions are governed by a linear constant-coefficient, nonhomogeneous system, whose solution is expressible in the form of an integral. Next, induction is employed to show that all the coefficient functions are bounded by functions having very simple forms. Finally, the comparison test and ratio test ensure the existence of a positive radius of convergence.

**Step one: Resummation of the above Psi-series**

Let \( \alpha = k + k_0 l \) and rewrite the series as

\[
x(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+l} \log \tau = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{k+l} \log \tau = \sum_{a=0}^{\infty} \sum_{k+k_0 l = a} a_{kl} \tau^{\alpha} \log \tau
\]

(2.5)

Let \( f_\alpha(z) = \sum_{k+k_0 l = a} a_{kl} (\log \tau)^l \),

(2.6)

then

\[
x(t) = \sum_{a=0}^{\infty} \tau^{\alpha} f_\alpha(z).
\]

Similarly,

\[
y(t) = \sum_{a=0}^{\infty} \tau^{\alpha} g_\alpha(z),
\]

(2.7)

where

\[
g_\alpha(z) = \sum_{k+k_0 l = a} b_{kl} (\log \tau)^l.
\]

(2.8)

**Step two: Change the Format of the System**

Let \( u_1 = x, \ u_2 = u_1', \ u_3 = u_2'; \ v_1 = y, \ v_2 = v_1, \ v_3 = v_2 \)

For example, with \( A = B = -1 \), the original system becomes

\[
\begin{align*}
u_1' &= u_2 \\
u_2' &= u_1 - 2Du_1 v_1 \\
v_1' &= v_2 \\
v_2' &= v_1 - Du_1^2 + Cv_1^2
\end{align*}
\]
Here \( u_i = \sum_{\alpha = 0}^{\infty} \tau^{\alpha-2} f_\alpha(z); \quad v_i = y(t) = \sum_{\alpha = 0}^{\infty} \tau^{\alpha-2} g_\alpha(z), \)

hence
\[
u_1' = u_2 = \sum_{\alpha = 0}^{\infty} [(\alpha-2)f_\alpha + f'_\alpha] \tau^{\alpha-3} = \sum_{\alpha = 0}^{\infty} h_\alpha \tau^{\alpha-3},
\]

where \( h_\alpha = (\alpha-2)f_\alpha + f'_\alpha, \)

\[
u_2' = \sum_{\alpha = 0}^{\infty} [(\alpha-3)h_\alpha + h'_\alpha] \tau^{\alpha-4},
\]

Substituting the above into the system and rearranging the terms, we get
\[
f'_\alpha + (\alpha-2)f_\alpha - h_\alpha = 0,
\]
\[
h'_\alpha + (\alpha-3)h_\alpha = Af_{\alpha-2} - 2D \sum_{\beta=0}^{\alpha} g_\alpha f_{\alpha-\beta},
\]
\[
g'_\alpha + (\alpha-2)g_\alpha - k_\alpha = 0,
\]
\[
k'_\alpha + (\alpha-3)k_\alpha = Bg_{\alpha-2} - D \sum_{\beta=0}^{\alpha} f_\alpha f_{\alpha-\beta} + C \sum_{\beta=0}^{\alpha} g_\alpha g_{\alpha-\beta}.
\]

Solving the above system with \( \alpha = 0 \) gives
\[
f_0 = \frac{\pm 3}{\lambda} \sqrt{2 + \frac{1}{\lambda}}, \quad g_0 = \frac{-3}{\lambda}.
\]

Now rewrite the above system as
\[
f'_\alpha + (\alpha-2)f_\alpha - h_\alpha = 0
\]
\[
h'_\alpha + 6f_\alpha + (\alpha-3)h_\alpha - 6 \sqrt{2 + \frac{1}{\lambda}} = Af_{\alpha-2} - 2D \sum_{\beta=0}^{\alpha} g_\alpha f_{\alpha-\beta}
\]
\[
g'_\alpha + (\alpha-2)g_\alpha - k_\alpha = 0
\]
\[
k'_\alpha + 6 \sqrt{2 + \frac{1}{\lambda}} + (\alpha-3)k_\alpha + \frac{6}{\lambda} g_\alpha = Bg_{\alpha-2} - D \sum_{\beta=0}^{\alpha} f_\alpha f_{\alpha-\beta} + C \sum_{\beta=0}^{\alpha} g_\alpha g_{\alpha-\beta}
\]
In matrix notation this is equivalent to

\[
\begin{vmatrix}
\alpha - 2 & -1 & 0 & 0 \\
-6 & \alpha - 3 & -6 \sqrt{2 + \frac{1}{\lambda}} & 0 \\
0 & 0 & \alpha - 2 & -1 \\
-6 \sqrt{\frac{1}{\lambda} + 2} & 0 & \frac{6}{\lambda} & \alpha - 3
\end{vmatrix}
\begin{vmatrix}
f_a \\
h_a \\
g_a \\
k_a
\end{vmatrix}
= 
\begin{vmatrix}
f_a \\
h_a \\
g_a \\
k_a
\end{vmatrix}
= \begin{vmatrix}
F_a \\
G_a
\end{vmatrix},
\]  
(2.9)

or

\[
f'_a + A_a f_a = F_a,
\]  
(2.10)

where

\[
\begin{vmatrix}
f_a \\
h_a \\
g_a \\
k_a
\end{vmatrix},
\quad A_a = \begin{vmatrix}
\alpha - 2 & -1 & 0 & 0 \\
-6 & \alpha - 3 & -6 \sqrt{2 + \frac{1}{\lambda}} & 0 \\
0 & 0 & \alpha - 2 & -1 \\
-6 \sqrt{\frac{1}{\lambda} + 2} & 0 & \frac{6}{\lambda} & \alpha - 3
\end{vmatrix}
\]

\[
\begin{vmatrix}
0 \\
0 \\
F_a \\
G_a
\end{vmatrix}
= \begin{vmatrix}
F_a \\
G_a
\end{vmatrix}
\]

and

\[
F_a = A f_{a-2} - 2D \sum_{\beta=1}^{a} g_\alpha f_{a-\beta},
\]

\[
G_a = B g_{a-2} - D \sum_{\beta=1}^{a} f_\alpha f_{a-\beta} + C \sum_{\beta=1}^{a} g_\alpha g_{a-\beta}.
\]

In our case, we have $\lambda = -1$, $A = B$, and take the negative $f_0$ to start, then $A_a$ becomes

\[
\begin{vmatrix}
\alpha - 2 & -1 & 0 & 0 \\
-6 & \alpha - 3 & -6 & 0 \\
0 & 0 & \alpha - 2 & -1 \\
-6 & 0 & -6 & \alpha - 3
\end{vmatrix}
\]

The eigenvalues of $A_a$ are found to be

\[
\xi_1 = \alpha + 1, \quad \xi_2 = \alpha - 2, \quad \xi_3 = \alpha - 3, \quad \xi_4 = \alpha - 6,
\]

so the eigenvalues are precisely $\alpha - r$, where $r$ is a resonance.
The corresponding eigenvectors are

\[
m_1 = \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ 1 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad m_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad m_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.
\]

hence \( A_\alpha \) is diagonalized as

\[
A_\alpha = P D_\alpha P^{-1},
\]

where \( P = [m_1, m_2, m_3, m_4] \),

\[
P^{-1} = \begin{bmatrix}
-6 & 3 & -6 & -3 \\
7 & 14 & 7 & 14 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 \\
6 & 2 & 6 & 2 \\
7 & 7 & 7 & 7
\end{bmatrix}, \quad D_\alpha = \begin{bmatrix}
\alpha + 1 & 0 & 0 & 0 \\
0 & \alpha - 2 & 0 & 0 \\
0 & 0 & \alpha - 3 & 0 \\
0 & 0 & 0 & \alpha - 6
\end{bmatrix}.
\]

The equation (2.10) is a first-order nonhomogeneous vector ODE whose solution, for \( \alpha \geq 6 \), is given by

\[
\vec{f}_\alpha (z) = \int_{-\infty}^{z} e^{A_\alpha (x-z)} \vec{F}_\alpha (x) \, dx,
\]

so

\[
\| \vec{f}_\alpha (z) \| \leq \int_{-\infty}^{z} \| e^{A_\alpha (x-z)} \| \| \vec{F}_\alpha (x) \| \, dx.
\]

For \( x \leq z \) and \( z \geq 6 \), we have the following

\[
\| \vec{F}_\alpha \| = |F_{\alpha}|,
\]

\[
\| e^{A_\alpha (x-z)} \| \leq \| P \| \| e^{D_\alpha (x-z)} \| \| P^{-1} \|,
\]

\[
= 6 \begin{bmatrix}
\left( e^{(\alpha-1)(x-z)} \right) & 0 & 0 & 0 \\
0 & \left( e^{(\alpha-2)(x-z)} \right) & 0 & 0 \\
0 & 0 & \left( e^{(\alpha-3)(x-z)} \right) & 0 \\
0 & 0 & 0 & \left( e^{(\alpha-6)(x-z)} \right)
\end{bmatrix}.
\]

Since \( x \leq z \), \( x - z \leq 0 \), so \( (\alpha - 6) < (\alpha - 3) < (\alpha - 2) < (\alpha + 1) \), and hence
\[ \|e^{\lambda z} \| \leq 6e^{(\alpha-6)(z^\alpha)}, \]  \hspace{2cm} (2.15)

so

\[ \|f_n(z)\| \leq 6 \int_{-\infty}^{\infty} e^{(\alpha-6)(x^\alpha)} |F_{\alpha}(x)| \, dx. \]  \hspace{2cm} (2.16)

We now state and prove a theorem regarding the convergence of the psi-series (2.3 and 2.4).

**Step three: The Convergence Proof**

**Theorem 2.1.** There exists \( k > 0 \) such that \( \forall z < 0 \)

\[ \|f_n(z)\| \leq \frac{(2k-kz)^\alpha}{\sqrt{\alpha+1}} \forall \alpha \geq 1. \]  \hspace{2cm} (2.17)

In order to prove the theorem, we need to state a Lemma proved by Melkonian and Hemmi [17].

**Lemma 2.1.** Let \( z \) be either positive or negative and let \( P_\alpha(z) \) be a sequence of polynomial functions of degree \( n_\alpha = \lceil \frac{\alpha}{q} \rceil \), \( q \) a positive integer, i.e.,

\[ P_\alpha(z) = \sum_{m=0}^{n_\alpha} c_m^{(\alpha)} z^m. \]  \hspace{2cm} (2.18)

Given an integer \( N > q \), \( 0 < M \leq 1 \), and \( p > 0 \), there exists a real constant \( K \), with sign \( (K) = \text{sign} (z) \), such that

\[ |P_\alpha(z)| \leq M^{\left\lfloor \frac{p|K|+Kz}{q} \right\rfloor} \frac{\alpha}{\sqrt{\alpha+1}}, \quad \text{for } 1 \leq \alpha \leq N-1 \]  \hspace{2cm} (2.19)

Now we are ready to prove the theorem.

**Proof:** Since \( f_\alpha, g_\alpha, k_\alpha \) are polynomials in \( z \) of degree \( \left\lfloor \frac{\alpha}{q} \right\rfloor \), by Lemma 1, with \( M = 1 \), \( q = 6, n_\alpha = \left\lfloor \frac{\alpha}{6} \right\rfloor \), \( p = 2 \), there exist four real positive constants \( k_f, k_g, k_n, k_k \) such that

\[ |f_\alpha| \leq \frac{(2k_f-kz)^\alpha}{\sqrt{\alpha+1}}, \quad |g_\alpha| \leq \frac{(2k_g-kz)^\alpha}{\sqrt{\alpha+1}}, \]
\[ |k_\alpha| \leq \frac{(2k_h - k_{h2}) \alpha}{\sqrt{\alpha + 1}}, \quad |n_\alpha| \leq \frac{(2k - k_{zz}) \alpha}{\sqrt{\alpha + 1}}, \]

for \( 1 \leq \alpha \leq N-1 \).

Let \( K = \max (k_h, k_g, k_h, k_k) \). Since \( \|f_\alpha(z)\| = \max (|f_\alpha(z)|, |g_\alpha(z)|, |h_\alpha(z)|, |k_\alpha(z)|) \), (2.19) holds for \( 1 \leq \alpha \leq N-1 \).

Let this be our inductive hypothesis, and let us prove that the result holds for \( \alpha = N \) and hence for any \( \alpha \geq 1 \).

Recall \( F_\alpha = A f_\alpha - 2D \sum_{\beta=1}^{\alpha} g_\alpha f_{\alpha-\beta} \),

\[ G_\alpha = B g_\alpha - D \sum_{\beta=1}^{\alpha} f_\alpha f_{\alpha-\beta} + D \sum_{\beta=1}^{\alpha} g_\alpha g_{\alpha-\beta}, \]

so

\[ |F_\alpha(z)| \leq |A| |f_\alpha| + 2D \sum_{\beta=1}^{\alpha-1} |g_\alpha| |f_{\alpha-\beta}| \leq |A| \left( \frac{(2k - k_{zz}) \alpha}{\sqrt{\alpha + 1}} \right) + 2D \left( \frac{(2k - k_{zz}) \alpha}{\sqrt{\alpha + 1}} \right) \sum_{\beta=1}^{\alpha-1} \frac{1}{\sqrt{\beta + 1} \sqrt{N - \beta + 1}} \]

\[ \leq \left( \frac{|A|}{\sqrt{N - 1}} + 2D \sum_{\beta=1}^{N-1} \frac{1}{\sqrt{\beta + 1} \sqrt{N - \beta + 1}} \right) \quad (2.20) \]

Now in order to further simplify the above results, we need to state a lemma proved by Melkonian and Hemmi [17].

**Lemma 2.2**: Let

\[ S(\alpha) = \sum_{\beta=0}^{\alpha-1} \frac{1}{\sqrt{\beta + 1} \sqrt{\alpha - \beta}}, \]

then \( \lim_{\alpha \to \infty} S(\alpha) = \pi \).

Now applying this Lemma, we have:

\[ \sum_{\beta=1}^{N-1} \frac{1}{\sqrt{\beta + 1} \sqrt{N - \beta + 1}} \leq \sum_{\beta=1}^{N-1} \frac{1}{\sqrt{\beta + 1} \sqrt{N - \beta}} \leq \pi \]

so (2.20) becomes

2-7
\[ |F_\alpha(z)| \leq \left( \frac{|A|}{\sqrt{N-1}} + 2|D|\pi \right) (2k-kz)^{\frac{N}{6}}, \quad (2.21) \]

Similarly, since \( \lambda = \frac{D}{C} = -1 \), so \( D = -C \),

\[ G_\alpha = Bg_{a-2} - D (\sum_{a=1}^{\alpha} f_{a} f_{a-\beta} + \sum_{a=1}^{\alpha} g_{a} g_{a-\beta}) \]
\[ \leq \left( \frac{|B|}{\sqrt{N-1}} + 2|D|\pi \right) (2k-kz)^{\frac{N}{6}}. \]

Let

\[ ||Q_\alpha|| = \begin{vmatrix} \frac{|A|}{\sqrt{N-1}} + 2|D|\pi & 0 \\ 0 & \frac{|B|}{\sqrt{N-1}} + 2|D|\pi \end{vmatrix}. \]

(Case 1) If \(|A| \geq |B|\) then \(||Q_\alpha|| = \left| \frac{|A|}{\sqrt{N-1}} + 2|D|\pi \right| . \]

(Case 2) If \(|B| > |A|\) then \(||Q_\alpha|| = \left| \frac{|B|}{\sqrt{N-1}} + 2|D|\pi \right| . \]

Therefore,

\[ |F_\alpha(z)| \leq ||Q_\alpha|| (2k-kz)^{\frac{N}{6}} \]

Substituting this estimate into (2.16) we get

(Case 1) \(|A| \geq |B|, \)

\[ ||f_\alpha(z)|| \leq 6 \left\{ \frac{|A|}{\sqrt{N-1}} + 2|D|\pi \right\} \tilde{e}^{(N-6)(N-2)} (2k-kz)^{\frac{N}{6}} dx \quad (2.22) \]

In order to estimate a lower bound for the integral in above we need to state another Lemma.

Lemma 2.3. Let \( z < 0, \ q \) a positive integer greater than or equal to 2, \( a > 0, \ b < 0, \ q > \frac{b}{a}, \ 0 < \delta < 1, \ \alpha > \frac{q}{1-\delta} \), and consider the integral
\[ I(z) = \int_{-\infty}^{\infty} e^{(\alpha-q)x^2} (a+bx)^{\alpha} \, dx. \]

Then
\[ I(z) \leq \frac{(a+bz)^{\alpha}}{\delta \alpha} (1 + \frac{\xi}{(q-\xi)\delta}) \], where \( \xi = \frac{b}{a} \).

The proof of the Lemma 2.3 was shown in Melkonian and Hemmi [17].

By Lemma 2.3, with \( a = 2k \), \( b = -k \), \( q = 6 \), \( \xi = \frac{1}{2} \), we have
\[ \int_{-\infty}^{\infty} e^{(1-\delta)x^2} (2k-kz)^{\frac{N}{8}} \, dx \leq \left( \frac{1+11\delta}{11N\delta^{2}} \right) \left( \frac{1+11\delta}{11N\delta^{2}} \right)^{\frac{N}{8}} \]

\[ \Rightarrow \quad \| f(z) \| \leq 6(\frac{|A|}{\sqrt{N-1}} + 2|D|\pi) \left( \frac{1+11\delta}{11N\delta^{2}} \right) \left( \frac{1+11\delta}{11N\delta^{2}} \right)^{\frac{N}{8}}. \]

We want (2.17) to hold for \( \alpha = N \). This is so if and only if \( N \) satisfies
\[ 6 \frac{\sqrt{N+1}}{N} \left( \frac{|A|}{\sqrt{N-1}} + 2|D|\pi \right) \leq \left( \frac{1+11\delta}{11N\delta^{2}} \right). \]  
(2.23)

Similarly for Case 2 where \( |B| > |A| \), if and only if \( N \) satisfies:
\[ 6 \frac{\sqrt{N+1}}{N} \left( \frac{|B|}{\sqrt{N-1}} + 2|D|\pi \right) \leq \left( \frac{1+11\delta}{11N\delta^{2}} \right), \]  
(2.24)

where \( 0 < \delta < 1 \). We also require that \( N \geq \frac{6}{1-\delta} \). We refer to equations (2.23) and (2.24) as the induction condition (i.e. the condition on \( N \) so that the induction proof works). Therefore, if we choose \( N \) sufficiently large so that for any larger \( N \) the induction condition is satisfied, we have that (2.11) holds for all \( \alpha \geq 1 \). Having found such \( N \), then \( k_f, k_g, k_n, k_k \) can easily be found and the theorem is proved.

Now we are ready to prove that the psi-series in (2.03) and (2.04) are convergent.

**Theorem 2.2.** There exists \( R, \ 0 < R < \infty \), such that the psi-series in (2.03) and (2.04) converge \( \forall \ \tau, \ 0 < \tau < R \)

**Proof:** By Theorem 1,
\[ |x(t)| = \sum_{\alpha=0}^{\infty} \tau^{-2} |f_\alpha(z)| \leq \sum_{\alpha=1}^{\infty} \frac{(2k-kz)^{\alpha}}{\sqrt{\alpha + 1}} \tau^{-2} \]

\[ |y(t)| = \sum_{\alpha=0}^{\infty} \tau^{-2} |g_\alpha(z)| \leq \sum_{\alpha=1}^{\infty} \frac{(2k-kz)^{\alpha}}{\sqrt{\alpha + 1}} \tau^{-2} \]

Applying the ratio test to the above two series, the two series converge if

\[(2K-Kz)^{1/6} < 1 \quad \text{or} \quad (2K-Kz) < e^{-6z}\]

It is clear that there exists a unique \( z_0 \) such that \( \forall z < z_0, (2K-Kz) < e^{-6z} \), hence by the comparison test, we have that the series converges absolutely for \( 0 < \tau < \epsilon \); and the proof is complete.
2.2 Complex or Irrational Conjugate Resonances

If the resonance is complex or irrational, Chang [11] found that the psi-series solution for the case (I) leading-orders takes the form of a double series,

\[
x(t) = t^{-2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{kj} \tau_0^k \tau_0^j + t^{-2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{kj} \tau_0^{-k} \tau_0^{-j}
\]

\[
y(t) = t^{-2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} b_{kj} \tau_0^k \tau_0^j + t^{-2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} b_{kj} \tau_0^{-k} \tau_0^{-j}
\]  

(2.25)  

(2.26)

where \( \tau_0 = t^\alpha \),  
\[
\alpha = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 24 \left( \frac{1}{\lambda} + 1 \right)},
\]

\[
\tau_0 = t^{-\alpha}, \quad \alpha = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 24 \left( \frac{1}{\lambda} + 1 \right)},
\]

and

\[
a_{00} = \pm \frac{3}{\lambda} \sqrt{2 + \frac{1}{\lambda}}, \quad b_{00} = \frac{-3}{\lambda}.
\]

For case (II) leading orders,

\[
x(t) = \tau_0 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{kj} \tau_0^k \tau_0^j + \tau_0^{-1} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{kj} \tau_0^{-k} \tau_0^{-j},
\]

\[
y(t) = t^{-2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} b_{kj} \tau_0^k \tau_0^j + t^{-2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} b_{kj} \tau_0^{-k} \tau_0^{-j},
\]  

(2.27)  

(2.28)

where

\[
\tau_0 = t^\alpha, \quad \alpha = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 48 \lambda},
\]

\[
\tau_0 = t^{-\alpha}, \quad \alpha = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 48 \lambda},
\]

and \( a_{00}, b_{00} \) are arbitrary, \( b_{00} = 6 \), and \( \tau_0 \) is the special case of \( \tau = (t-t_0)^a \) with \( t_0 = 0 \).

Note 1. In their paper [11], Y.F. Chang et al. have slightly different forms for \( x(t) \) and \( y(t) \) from the above (2.21 - 2.23); for their double summation in \( x(t) \) and \( y(t) \), their index \( k \) starts from one. By adding some extra zeroes, i.e. we assume \( a_{0j} = 0 \), we are able to rewrite the double series \( x(t) \) and \( y(t) \) in the above format. The purpose of doing this is to make our analysis cleaner.
Note 2. The above series is slightly different form what Melkonian and Hemmi [17] stated for their complex resonance case. They stated that the psi-series for the case of complex-conjugate resonances are of the form

$$u = \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} a_{\ell j} \tau^{\ell} (\tau^*)^j + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \tilde{a}_{\ell j} \tau^{\ell} (\tilde{\tau}^*)^j,$$  \hspace{1cm} (2.29)

where \( r \) and \( \tilde{r} \) are two complex conjugate resonances.

In the above series, for the case (I) leading order, \( \alpha = \frac{1}{2} + \frac{1}{2 \sqrt{1 - 24 (\frac{1}{\lambda} + 1) }} \) and

\[
\tilde{\alpha} = \frac{1}{2} - \frac{1}{2 \sqrt{1 - 24 (\frac{1}{\lambda} + 1) }}
\]

are complex conjugates, but they are not resonances. The relationship between the two is \( r = 2 + \alpha \). Could we change the above \( x(t) \) and \( y(t) \) into Melkonian and Hemmi's format? In other words, if we change \( \alpha \) into \( r \) in the above, will it still be a solution? It turns out that if we change variables by letting (Chang et al [11])

\[
x(t) = \frac{1}{t^2} \theta(X), \hspace{1cm} y(t) = \frac{1}{t^2} \psi(X),
\]

where \( X = t^{\alpha-2} \), \( \alpha = \frac{1}{2} + \frac{1}{2 \sqrt{1 - 24 (\frac{1}{\lambda} + 1) }} \), the new series \( \theta(X) \) and \( \psi(X) \) are also solutions for the system; they give the system the same recursion relationships, and it displays exactly the same sort of singularities in the \( X \)-plane as do \( x(t) \) and \( y(t) \) in the \( t \)-plane. Therefore, close to a given singularity, the singularity structure of \( x(t) \) and \( y(t) \) may be determined by studying the singularities of \( \theta(X) \) and \( \psi(X) \). The key is to correctly map the singularities from the \( X \)-plane to the \( t \)-plane. Now \( \theta(X) = \tilde{r}^2 x(t) \) which is the same format as (2.29) in Melkonian and Hemmi[17], and we have \( \alpha = \frac{1}{2} + \frac{1}{2 \sqrt{1 - 24 (\frac{1}{\lambda} + 1) }} \), which is the complex resonance in the psi-series. Later on, we will use Melkonian and Hemmi's format to do the convergence proof for the above psi-series.

Similarly for Case (II), by letting

\[
x(t) = t^{\alpha} \theta(X) \hspace{1cm} \text{and} \hspace{1cm} y(t) = \frac{1}{t^2} \psi(X), \hspace{1cm} \text{where} \hspace{1cm} \alpha = \frac{1}{2} + \frac{1}{2 \sqrt{1 - 48 \lambda }},
\]

we get the Melkonian and Hemmi's format for Case (II).

In this paper, unless otherwise stated, we shall use Melkonian and Hemmi's format for the convergence proof of the Case (I) leading order.
First, we need to check if the above psi-series yield enough free constants. If we substitute \( x(t) \) and \( y(t) \) back into the H-H system, the recursion relation is found to be

\[
(\alpha k - j - 2) (\alpha k + j - 3) a_{kj} + a_{kj - 2} + 2\lambda \sum_{l=0}^{k} \sum_{m=0}^{j} a_{k+l, j-m} b_{lm}
+ 2\lambda \sum_{n=0}^{j} \sum_{m=0}^{j-n} (a_{k+n, m} b_{n, j-n-m} + b_{k-n, m} a_{n, j-n-m}) = 0,
\]

(2.30)

\[
(\alpha k + j - 2) (\alpha k + j - 3) a_{kj} + a_{kj - 2} + 2\lambda \sum_{l=0}^{k} \sum_{m=0}^{j} a_{k+l, j-m} b_{lm}
+ 2\lambda \sum_{n=0}^{j} \sum_{m=0}^{j-n} (a_{k+n, m} b_{n, j-n-m} + b_{k-n, m} a_{n, j-n-m}) = 0,
\]

(2.31)

\[
(\alpha k + j - 2) (\alpha k + j - 3) b_{kj} + b_{kj - 2} + \sum_{l=0}^{k} \sum_{m=0}^{j} (\lambda a_{k+l, j-m} a_{lm} - b_{k+l, j-m} b_{lm})
+ \sum_{n=0}^{j} \sum_{m=0}^{j-n} (\lambda a_{k+n, m} a_{n, j-n-m} - b_{k+n, m} b_{n, j-n-m}) = 0.
\]

(2.32)

\[
(\alpha k + j - 2) (\alpha k + j - 3) b_{kj} + b_{kj - 2} + \sum_{l=0}^{k} \sum_{m=0}^{j} (\lambda a_{k+l, j-m} a_{lm} - b_{k+l, j-m} b_{lm})
+ \sum_{n=0}^{j} \sum_{m=0}^{j-n} (\lambda a_{k+n, m} a_{n, j-n-m} - b_{k+n, m} b_{n, j-n-m}) = 0.
\]

(2.33)

It can be verified (Chang et al.) that the expansions defined by equations (2.30 - 2.33) are consistent and well defined. The arbitrary constants are \( b_{06}, a_{12} \) and \( \alpha_{12} \). So the compatibility conditions are met.

Similarly, for the Case (II) leading order, the recursion relations are found to be

\[
[\alpha (k+1) + j] [\alpha (k+1) + j - 1] a_{kj} + a_{kj - 2} + 2\lambda \sum_{l=0}^{k} \sum_{m=0}^{j} a_{k+l, j-m} b_{lm}
+ 2\lambda \sum_{n=0}^{j} \sum_{m=0}^{j-n} (a_{k+n, m} b_{n, j-n-m} + b_{k-n, m} a_{n, j-n-m}) = 0,
\]

(2.34)

\[
[\alpha (k+1) + j][\alpha (k+1) - j] a_{kj} + a_{kj - 2} + 2\lambda \sum_{l=0}^{k} \sum_{m=0}^{j} a_{k+l, j-m} b_{lm}
\]

2-13
+ 2\lambda \sum_{m=0}^{j-n} \sum_{n=0}^{j-n} (a_{k-n, m} b_{n, j-n-m} + b_{k-n, m} a_{n, j-n-m}) = 0, \quad (2.35)

(\alpha k + j - 2) (\alpha k + j - 3) b_k + b_{k-2} + \lambda \sum_{l=0}^{k-2} \sum_{m=0}^{j-4} a_{k-3, j-4-m-2} a_{l, m} - \sum_{l=0}^{k} \sum_{m=0}^{j} b_{k-l, j-m} b_{l, m}

+ \lambda \sum_{m=0}^{j-5} \sum_{n=0}^{j-5} a_{k-n, m} a_{n, j-n-m} b_{n, j-n-m} = 0, \quad (2.36)

(\alpha k + j - 2)(\alpha k + j - 3) b_{k-j} + b_{k-2} + \lambda \sum_{l=0}^{k-2} \sum_{m=0}^{j-4} a_{k-2, j-4-m-2} a_{l, m} - \sum_{l=0}^{k} \sum_{m=0}^{j} b_{k-l, j-m} b_{l, m}

+ \lambda \sum_{m=0}^{j-5} \sum_{n=0}^{j-5} a_{k-n, m} a_{n, j-n-m} b_{n, j-n-m} = 0. \quad (2.37)

The above expansions are checked to be consistent and yield free constants \(a_{00}, \ a_{00}, \ b_{06}\). Now we need to give the convergence proof for the above psi-series. We shall consider the two cases separately in Chapters 3 and 4.
Chapter 3

Convergence Proofs For Psi-Series

Consider the psi-series with the complex or irrational conjugate resonances, in order to show they are the actual solutions of the H-H system, we need to prove that they are convergent. The proofs of convergence are similar to the proof of the psi-series in Chapter 2. The convergence proof consists of the same four steps, namely the multiple series is resummed into single series with function coefficients, then it is shown that the coefficient functions are governed by a linear, constant-coefficient, nonhomogeneous system of ODEs, whose solution is expressible in the form of an integral; next, induction is employed to show that all of the coefficient functions are bounded by functions having very simple forms. Finally, the comparison test and the ratio test ensure the existence of a positive number $\mathcal{R}$ such that the psi-series under consideration converges for $0 < \tau < \mathcal{R}$. The main difference between the two cases is in the resummation of the psi-series.

3.1 Case (I) - Convergence Proof of the H-H System

First we need to do a resummation of the above psi-series. It can be resummed into the form [17]

\begin{align*}
    x(t) &\equiv u_I = \sum_{\alpha=0}^{\infty} f_\alpha(z) \tau^{\alpha-2}, \\
    y(t) &\equiv v_I = \sum_{\alpha=0}^{\infty} g_\alpha(z) \tau^{\alpha-2},
\end{align*}

(3.1) (3.2)

where $z = \log \tau$ and the functions $f_\alpha(z)$ and $g_\alpha(z)$ are given by

\begin{align*}
    f_\alpha(z) &= \sum_{n=0}^{\alpha} \left[ a_n^\alpha \cos(n\mu z) + b_n^\alpha \sin(n\mu z) \right] e^{n(\nu-1)z}, \\
    g_\alpha(z) &= \sum_{n=0}^{\alpha} \left[ c_n^\alpha \cos(n\mu z) + d_n^\alpha \sin(n\mu z) \right] e^{n(\nu-1)z},
\end{align*}

and \[ \mu = \frac{\sqrt{1-24(\frac{1}{\lambda}+1)}}{2}, \quad \nu = \frac{5}{2}. \]

Similar to the case which has incompatible integer resonances, putting (3.1) and (3.2) into the H-H system and arranging some terms, we get the following system

3-1
\[
\begin{bmatrix}
 f_a \\
 h_a \\
 g_a \\
 k_a 
\end{bmatrix} + \begin{bmatrix}
 \alpha - 2 & -1 & 0 & 0 \\
 -6 & \alpha - 3 & -6\sqrt{2 + \frac{1}{\lambda}} & 0 \\
 0 & 0 & \alpha - 2 & -1 \\
 -6\sqrt{2 + \frac{1}{\lambda}} & 0 & \frac{6}{\lambda} & \alpha - 3 
\end{bmatrix} \begin{bmatrix}
 f_a \\
 h_a \\
 g_a \\
 k_a 
\end{bmatrix} = \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 G_a 
\end{bmatrix}, \tag{3.3}
\]

Write the above as
\[
\vec{f}_a + A_\alpha \vec{f}_a = \vec{F}_a,
\]
where \( F_a = A f_{a-2} - 2D \sum_{\beta=1}^{a-1} g_\alpha f_{a-\beta} \),
\[G_a = B g_{a-2} - D \left[ \sum_{\beta=1}^{a-1} f_\alpha f_{a-\beta} + \sum_{\beta=1}^{a} g_\alpha g_{a-\beta} \right].\]

The eigenvalues of \( A_\alpha \) are found by solving
\[
\det \begin{bmatrix}
 \alpha - 2 - \xi & -1 & 0 & 0 \\
 -6 & \alpha - 3 - \xi & -6\sqrt{2 + \frac{1}{\lambda}} & 0 \\
 0 & 0 & \alpha - 2 - \xi & -1 \\
 -6\sqrt{2 + \frac{1}{\lambda}} & 0 & \frac{6}{\lambda} & \alpha - 3 - \xi 
\end{bmatrix} = 0
\]

Let \( \alpha - \xi = u \) to get
\[
u^4 - 10u^3 + \left[ 30 + \frac{6}{\lambda} \right] u^2 - 30\left( 1 + \frac{1}{\lambda} \right) u - 36\left( 2 + \frac{1}{\lambda} \right) = 0. \tag{3.4}
\]

Solving the equation, the corresponding eigenvalues are found to be
\[
\xi_1 = \alpha + 1, \quad \xi_2 = \alpha - 6, \quad \xi_3 = \alpha - \left( \frac{5}{2} + \frac{1}{2} \sqrt{1 - 24\left( \frac{1}{\lambda} + 1 \right)} \right),
\]
and \( \xi_4 = \alpha - \left( \frac{5}{2} - \frac{1}{2} \sqrt{1 - 24\left( \frac{1}{\lambda} + 1 \right)} \right). \)
The corresponding eigenvectors are found to be

\[
\begin{align*}
V_1 &= \frac{-\sqrt{2 + \frac{1}{\lambda}}}{3}, \quad V_2 = \frac{\sqrt{2 + \frac{1}{\lambda}}}{4}, \quad V_3 = \frac{-1 + \sqrt{1 - 24(\frac{1}{\lambda} + 1)}}{(1 + \frac{1}{\lambda})\sqrt{2 + \frac{1}{\lambda}}} \frac{1}{2 + \frac{1}{\lambda}}, \quad V_4 = \frac{-1 - \sqrt{1 - 24(\frac{1}{\lambda} + 1)}}{(1 + \frac{1}{\lambda})\sqrt{2 + \frac{1}{\lambda}}} \\
\end{align*}
\]

so the matrix of eigenvectors \( P \) of \( A_{\alpha} \) is given by

\[
P = \begin{pmatrix}
-\sqrt{2 + \frac{1}{\lambda}} & \frac{\sqrt{2 + \frac{1}{\lambda}}}{4} & -1 + \sqrt{1 - 24(\frac{1}{\lambda} + 1)} & -1 - \sqrt{1 - 24(\frac{1}{\lambda} + 1)} \\
\frac{-1}{3} & \frac{1}{4} & \frac{1}{2 + \frac{1}{\lambda}(1 + \frac{1}{\lambda})} & \frac{-1}{2 + \frac{1}{\lambda}(1 + \frac{1}{\lambda})} \\
\sqrt{2 + \frac{1}{\lambda}} & \sqrt{2 + \frac{1}{\lambda}} & -\frac{1}{\sqrt{2 + \frac{1}{\lambda}}} & -\frac{1}{\sqrt{2 + \frac{1}{\lambda}}} \\
\frac{-1}{3} & \frac{-1}{4} & 1 - \sqrt{1 - 24(\frac{1}{\lambda} + 1)} & 1 + \sqrt{1 - 24(\frac{1}{\lambda} + 1)} \end{pmatrix}
\]

1 1 1 1
$P^{-1}$ is given by

\[
\begin{align*}
&-12 \frac{\sqrt{2+\frac{1}{\lambda}}}{7} (1+\frac{1}{\lambda}) & 3 \frac{\sqrt{2+\frac{1}{\lambda}}}{7} (3+\frac{1}{\lambda}) & 12 \frac{1}{7} (1+\frac{1}{\lambda}) & 3 \frac{1}{7} (3+\frac{1}{\lambda}) \\
&12 \frac{\sqrt{2+\frac{1}{\lambda}}}{7} (1+\frac{1}{\lambda}) & 4 \frac{\sqrt{2+\frac{1}{\lambda}}}{7} (3+\frac{1}{\lambda}) & -12 \frac{1}{7} (1+\frac{1}{\lambda}) & 4 \frac{1}{7} (3+\frac{1}{\lambda}) \\
&(-1-\sqrt{1-24(\frac{1}{\lambda}+1)})\sqrt{2+\frac{1}{\lambda}} & -3\sqrt{2+\frac{1}{\lambda}}(2+\frac{1}{\lambda}) & -\frac{7}{24} \left(1+\sqrt{1-24(\frac{1}{\lambda}+1)}\right) & 2+\frac{1}{\lambda} \\
&2(1+\frac{1}{\lambda}) & 3+\frac{1}{\lambda} & \frac{24}{24} \left(1+\sqrt{1-24(\frac{1}{\lambda}+1)}\right) & 3+\frac{1}{\lambda} \\
&(-1-\sqrt{1-24(\frac{1}{\lambda}+1)})\sqrt{2+\frac{1}{\lambda}} & 0 & -\frac{7}{24} \left(1+\sqrt{1-24(\frac{1}{\lambda}+1)}\right) & 0 \\
&2(1+\frac{1}{\lambda}) & 0 \\
\end{align*}
\]

$D_{\alpha}$ is given by

\[
D_{\alpha} = P^{-1} A_{\alpha} P
\]

\[
D_{\alpha} = \begin{bmatrix}
\alpha + 1 & 0 & 0 & 0 \\
0 & \alpha - 6 & 0 & 0 \\
0 & 0 & \alpha - (\frac{5}{2} + \frac{\sqrt{1-24(\frac{1}{\lambda}+1)}}{2}) & 0 \\
0 & 0 & 0 & \alpha - (\frac{5}{2} - \frac{\sqrt{1-24(\frac{1}{\lambda}+1)}}{2})
\end{bmatrix}
\]

Now $e^{D_{\alpha}(x-z)}$ can be written as

\[
e^{D_{\alpha}(x-z)} = P e^{D_{\alpha}(x-z)} P^{-1}
\]
In order to solve the above first-order nonhomogenous vector ODE for $\alpha \geq 2$, we multiply by $e^{A_\alpha(x)}$ on both sides of the above system and then integrate both sides to get

$$ \vec{f}_\alpha(x) = \int_{-\infty}^{x} P e^{D_{\alpha(x-2)}} P^{-1} \vec{F}_\alpha(x) \, dx. $$

(3.5)

In order to estimate $\|f_\alpha(x)\|$, we need to estimate $\|P\|$, $\|P^{-1}\|$ and $\|e^{D_{\alpha(x-2)}}\|$. 

According to the definition, for a matrix $A$,

$$ \|A\| = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} |a_{ij}| \right), $$

and

$$ \|P\| = \max \left\{ 4, 2\sqrt{2 + \frac{1}{\lambda}}, \frac{1}{12} \sqrt{2 + \frac{1}{\lambda}}, \frac{2\sqrt{24(1 + \frac{1}{\lambda})}}{|1 + \frac{1}{\lambda}|}, \frac{7}{12} + \frac{\sqrt{24(1 + \frac{1}{\lambda})}}{6|1 + \frac{1}{\lambda}|} \right\}, $$

$$ \|P^{-1}\| = \max \left\{ \frac{12(\sqrt{2 + \frac{1}{\lambda}} + 1)}{7|1 + \frac{1}{\lambda}|} + \frac{3(\sqrt{2 + \frac{1}{\lambda}} + 1)}{7|3 + \frac{1}{\lambda}|}, \frac{\sqrt{24(1 + \frac{1}{\lambda})(2 + \frac{1}{\lambda})}}{2|1 + \frac{1}{\lambda}|} + \frac{|(2\lambda + 1)(\sqrt{2 + \frac{1}{\lambda}} + 1)|}{|(3\lambda + 1)|}, \frac{\frac{1}{24}|1 + \frac{1}{\lambda}|}{24|1 + \frac{1}{\lambda}|} \right\}. $$

Let $C = \|P\| \|P^{-1}\|$. Then $C$ is a constant depending on the value of $\lambda$. For example, when $\lambda < -\frac{24}{23}$, then we have

$$ \|P\| = 2\sqrt{2 + \frac{1}{\lambda}} + \frac{2}{\sqrt{2 + \frac{1}{\lambda}}}, $$

3-5
\[
\|P^{-1}\| = \sqrt{\frac{24(1 + \frac{1}{\lambda})(2 + \frac{1}{\lambda})}{2\lambda + \frac{1}{\lambda}}} + \frac{|(2\lambda + 1)|\left(\frac{3}{2}\sqrt{2 + \frac{1}{\lambda}} + 1\right)}{|(3\lambda + 1)|} + \frac{7\sqrt{1 - 24(\frac{1}{\lambda} + 1)}}{24\lambda + \frac{1}{\lambda}}.
\]

Now
\[
e^{A_d(x-z)} = P e^{D_d(x-z)} P^{-1}
\]

where
\[
e^{D_d(x-z)} = \begin{bmatrix}
e^{(\alpha+1)(x-z)} & 0 & 0 & 0 \\
0 & e^{(\alpha-6)(x-z)} & 0 & 0 \\
0 & 0 & e^{(\alpha-\nu+iu)} & 0 \\
0 & 0 & 0 & e^{(\alpha-\nu-iu)}
\end{bmatrix}
\]

Since \(\alpha > 6 > \nu = \frac{5}{2}\), which gives
\[
\alpha - 6 < \alpha - \nu < \alpha - 1,
\]

and since \(x-z < 0\),

\[
\|e^{D_d(x-z)}\| = \max \{ e^{(\alpha+1)(x-z)} , e^{(\alpha-6)(x-z)} , e^{(\alpha-\nu)(x-z)} \}
\]

For all \(\alpha > 6\),

\[
\|f_a(x)\| \leq C \int_{-\infty}^{\infty} e^{(\alpha-6)(x-z)} |F_a| \, dx
\]

\[
= C \int_{-\infty}^{\infty} e^{(\alpha-6)(x-z)} |F_a(x)| \, dx \tag{3.6}
\]

where \(C\) is a constant dependent upon \(\lambda\).

We now state and prove a theorem regarding the convergence of the psi-series for Case (I).

**Theorem 3.1.** There exists a positive number \(R\) such that the psi-series converges for all \(\tau, \ 0 < \tau < R\).

**Proof:**

We begin by the following

Claim: There exists a positive number \(K\) such that
\[ \|f_\alpha(z)\| \leq \frac{(11cK + K(e^{(v-1)z})^\alpha)}{\sqrt{\alpha + 1}} \quad \forall \alpha \geq 1, \quad z < 0 \]  

(3.7)

Proof of the claim:

From the above we have:

\[ f_\alpha(z) = \sum_{n=0}^{\alpha} \left[ a_n^\alpha \cos(n\mu z) + b_n^\alpha \sin(n\mu z) \right] e^{n(v-1)z} \]

\[ g_\alpha(z) = \sum_{n=0}^{\alpha} \left[ c_n^\alpha \cos(n\mu z) + d_n^\alpha \sin(n\mu z) \right] e^{n(v-1)z} \]

where \( \mu = \sqrt{1 - 24(\frac{1}{\lambda} + 1)} \)

Since \( |\cos(n\mu z)| \leq 1 \) and \( |\sin(n\mu z)| \leq 1 \), we have

\[ |f_\alpha(z)| \leq \sum_{n=0}^{\alpha} |C_n^\alpha| e^{n(v-1)z}, \]

\[ |g_\alpha(z)| \leq \sum_{n=0}^{\alpha} |D_n^\alpha| e^{n(v-1)z}, \]

where \( C_n^\alpha = a_n^\alpha + b_n^\alpha \)
\( D_n^\alpha = c_n^\alpha + d_n^\alpha \).

The superscripts \((\alpha)\) simply indicate dependence upon the \( f_\alpha \) and \( g_\alpha \), and \( C_n^\alpha \) and \( D_n^\alpha \) are constants. Since \( k_\alpha(z) \) and \( h_\alpha(z) \) are linear combinations of \( f_\alpha \) and \( g_\alpha \) and their derivatives, both \( |k_\alpha(z)| \) and \( |h_\alpha(z)| \) are dominated in the same manner as \( |f_\alpha| \) and \( |g_\alpha| \).

By Lemma 2.1, with \( M = 1 \), \( n_\alpha = \alpha \), \( p > 0 \), there exist four real positive constants, \( k_f \), \( k_g \), \( k_h \), \( k_k \), such that for \( 1 \leq \alpha \leq N-1 \),

\[ |f_\alpha(z)| \leq \frac{(pk_f + k_f e^{(v-1)z})^\alpha}{\sqrt{\alpha + 1}}, \quad |g_\alpha(z)| \leq \frac{(pk_g + k_g e^{(v-1)z})^\alpha}{\sqrt{\alpha + 1}}, \]

\[ |h_\alpha(z)| \leq \frac{(pk_h + k_h e^{(v-1)z})^\alpha}{\sqrt{\alpha + 1}}, \quad |k_\alpha(z)| \leq \frac{(pk_k + k_k e^{(v-1)z})^\alpha}{\sqrt{\alpha + 1}}. \]
Therefore by definition,

\[ \| f_\alpha(z) \| \leq \frac{(pk + ke^{(v-1)z})^\alpha}{\sqrt{\alpha + 1}} \]

\[ 0 \leq \alpha \leq N-1 , \]

where \( k = \max (k_1, k_2, k_3, k_4, k_5) \).

Let this be our inductive hypotheses, and let us prove that with the same \( k \), the result is true for \( \alpha = N \). First, we need to estimate \( |F_N(z)| \). For \( N \geq 7 \), we have

\[ |F_N(z)| \leq Af_N^\alpha \leq \frac{A}{\sqrt{N-1}} (pk + ke^{(v-1)z})^{(N-1)/2} + \frac{2D}{\sqrt{N-1}} \sum_{\beta=1}^{N-1} \frac{1}{\sqrt{\beta + 1} \sqrt{N - \beta + 1}} \]

\[ \leq \left[ \frac{A}{\sqrt{N-1}} + 2D\pi \right] (pk + ke^{(v-1)z})^N \]

Substitution of the latter into (3.5) \( (\alpha = N) \) gives:

\[ \| f_N(z) \| \leq C\left( \frac{A}{\sqrt{N-1}} + 2D\pi \right) \int_\infty^z e^{\left(N-\delta\right)(x^\alpha)} (pk + ke^{(v-1)z})^N \, dx . \] (3.8)

In order to get an estimate of the above integral, we need to state a lemma.

**Lemma 3.1.** Let \( \alpha > \delta, v > 1, a > 0, b > 0, \frac{b}{a} < \frac{1}{2(v - 1)}, z < 0 \), \( \alpha \) is an integer and consider the integral

\[ I(z) = \int_\infty^z (a + be^{(v-1)x})^\alpha e^{(a-\delta)(x^\alpha)} \, dx \] (3.9)

Then

\[ I(z) \leq \frac{(a + be^{(v-1)z})^\alpha}{(\alpha - \delta)(1 - 2(v - 1)\xi)} , \]

where \( \xi = \frac{b}{a} \).

This lemma is a generalized version of a lemma that was stated and proved by Melkonian and Hemmi [17]. The proof of the lemma is the same as it is in [17] except that we replace \( \alpha - \nu \) with \( \alpha - \delta \).

With \( a = pK, b = K, \alpha = N, \delta = 6, p = 11 \), we apply the above lemma to the integral

\[ I(z) = \int_\infty^z e^{(N-6)(x^\alpha)} (pk + ke^{(v-1)x})^N \, dx . \] (3.10)

to obtain

3-8
\[ I(z) \leq \frac{(11K + Ke^{(v-1)z})^N}{(N-6)(1 - 2(v-1)\xi)} , \quad \xi = \frac{b}{a} = \frac{1}{11}. \]

Substituting into (3.8), we get

\[ \| \tilde{f}_N(z) \| \leq 11C\left( \frac{A}{\sqrt{N-1}} + 2D\pi \right) \left( \frac{(11K + Ke^{(v-1)z})^N}{N-6} \right) . \] \hspace{1cm} (3.11)

Hence the claim holds for \( \alpha = N \), if and only if \( N \) satisfies:

\[ 11C \left( \frac{A}{\sqrt{N-1}} + 2D\pi \right) \frac{\sqrt{N+1}}{N-6} \leq 1 . \] \hspace{1cm} (3.12)

This is our induction condition and is satisfied for all sufficiently large \( N \).

We now return to the proof of the theorem. Since for all \( \alpha \geq 1 \), \( |f_\alpha(z)| \leq \| \tilde{f}_\alpha(z) \| \).

we have

\[ \sum_{\alpha=1}^\infty |f_\alpha(z)| \tau^{\alpha-2} \leq \sum_{\alpha=1}^\infty \frac{(11K + Ke^{(v-1)z})^\alpha}{\sqrt{\alpha+1}} \tau^{\alpha-2} . \] \hspace{1cm} (3.13)

By the ratio test, the series is convergent if

\[ (11K + Ke^{(v-1)z}) |\tau| < 1 \text{ or } 11K + Ke^{(v-1)z} < e^\tau . \] \hspace{1cm} (3.14)

This is the condition for the existence of a positive radius of convergence. Equality holds in (3.14) for a unique negative \( z_0 \) such that for all \( z < z_0 \) (3.12) is true. This implies that the series in (3.13) converges for \( 0 < \tau < e^{z_0} \). Therefore the solution converges absolutely for \( 0 < \tau < R \), where the radius of convergence is at least \( e^{z_0} \). This completes the proof of the theorem.
3.2 Case (II) - The Convergence Proof of The H-H system.

From Chapter 2, the solutions for Case (II) are of the form

\[ x(\tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} \tau^{\frac{k}{2} - \alpha} (\tau^*)^l + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \tilde{a}_{kl} \tau^{\frac{k}{2} - \rho} (\tau^*)^l, \quad (3.15) \]

\[ y(\tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} \tau^{\frac{k}{2} + \beta} (\tau^*)^l + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \tilde{b}_{kl} \tau^{\frac{k}{2} + \rho} (\tau^*)^l, \quad (3.16) \]

where \( \alpha \) and \( \beta \) are the leading orders for \( x \) and \( y \), in this case \( \alpha = \frac{1}{2} + \frac{1}{2} \sqrt{1-48\lambda} \). 
\( \beta = -2, \ r = \frac{1}{2} + \frac{1}{2} \sqrt{1-48\lambda} \), and \( \tilde{a}_{kl} \) and \( \tilde{b}_{kl} \) are complex conjugates of \( a_{kl} \) and \( b_{kl} \), respectively.

For this paper, we consider only the case \( \alpha = \frac{1}{2} + \frac{1}{2} \sqrt{1-48\lambda} \); the other case is similar. 

We break the convergence proof into several steps.

**Step 1: Resummation of the psi-series**

Write \( r = \nu + i \mu \), where \( \nu = \frac{1}{2} \) and \( \mu = \frac{1}{2} \sqrt{1-48\lambda} \), so that

\[ x(\tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (a_{kl} \tau^{\nu l} + \tilde{a}_{kl} \tau^{\mu l}) \tau^{\nu l} \tau^{\mu l}. \quad (3.17) \]

Express \( a_{kl}, \tilde{a}_{kl}, \tau^{\nu l}, \tau^{\mu l} \) as

\[ a_{kl} = a_{kl}^{re} + ia_{kl}^{im}, \quad \tau^{\nu l} = \cos(\mu z) + i \sin(\mu z), \]

\[ \tilde{a}_{kl} = a_{kl}^{re} - ia_{kl}^{im}, \quad \tau^{\mu l} = \cos(\mu z) - i \sin(\mu z), \]

where \( a_{kl}^{re} \) and \( a_{kl}^{im} \) are the real and imaginary parts of \( a_{kl} \), respectively, and let \( z = \log \tau \). 

The quantity between parentheses in (3.17) then becomes

\[ a_{kl} \tau^{\nu l} + \tilde{a}_{kl} \tau^{\mu l} = (a_{kl} + \tilde{a}_{kl}) \cos(\mu z) + i (a_{kl} - \tilde{a}_{kl}) \sin(\mu z) = 2 a_{kl}^{re} \cos(\mu z) - 2a_{kl}^{im} \sin(\mu z). \]
Substituting the latter into (3.17) yields

$$x(\tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[ A_{kl} \cos(\mu l \omega) + B_{kl} \sin(\mu l \omega) \right] \tau^l \tau^{k-p},$$

(3.18)

where $A_{kl} = 2a^l = 2 \text{Re}(a_{kl})$ and $B_{kl} = -2a^l = -2 \text{Im}(a_{kl})$, and writing

$$\tau^{k-p} \tau^l = \tau^{k-l} \tau^{(n-1)l},$$

where $n$ is the least integer such that $0 < \frac{l}{n} < 1$.

Then equation (3.18) becomes

$$x(\tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[ A_{kl} \cos(\mu l \omega) + B_{kl} \sin(\mu l \omega) \right] \tau^{k-l} \tau^{(n-1)l}.$$

Let $\alpha = k + \frac{l}{n}$ to obtain

$$x(\tau) = \sum_{\alpha=0}^{\infty} \left( \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[ A_{kl} \cos(\mu l \omega) + B_{kl} \sin(\mu l \omega) \right] \tau^{(n-1)l} \right) \tau^{\alpha-p}.$$

(3.19)

Since for each $\alpha$ there are a finite number of pairs $(k,l)$ such that $k + \frac{l}{n} = \alpha$, the inner sum in the above is finite and can be written as

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[ A_{kl} \cos(\mu l \omega) + B_{kl} \sin(\mu l \omega) \right] \tau^{(n-1)l} = \sum_{m=0}^{\infty} \left( a_m^{(a)} \cos(\mu m \omega) + b_m^{(a)} \sin(\mu m \omega) \right) e^{m(\omega-1)\omega},$$

where $a_m^{(a)} = A_{a-m,m} = 2 \text{Re}(a_{a-m,m})$ and $b_m^{(a)} = B_{a-m,m} = -2 \text{Im}(a_{a-m,m})$. Now define the function $f_\alpha(z)$ by

$$f_\alpha(z) = \sum_{m=0}^{\infty} \left( a_m^{(a)} \cos(\mu m \omega) + b_m^{(a)} \sin(\mu m \omega) \right) e^{m(\omega-1)\omega},$$

so that the psi-series for Case (II) converts into the single series

$$x(\tau) = \sum_{\alpha=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[ A_{kl} \cos(\mu l \omega) + B_{kl} \sin(\mu l \omega) \right] \tau^{k+\alpha} \left( \tau \tau^{(n-1)l} \right) = \sum_{\alpha=0}^{\infty} f_\alpha(z) \tau^{\alpha-p}$$

(3.20)

Similarly, for $y(\tau)$, we have

$$g_\alpha(z) = \sum_{m=0}^{\infty} \left( c_m^{(a)} \cos(\mu m \omega) + d_m^{(a)} \sin(\mu m \omega) \right) e^{m(\omega-1)\omega},$$

(3.21)

and
\[ y(\tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} \tau^{k+\theta}(\tau^\gamma)^l + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} \tau^k \psi (\tau) \tau^l = \sum_{a=0}^{\infty} g_a(z) \tau^{a-p} \]  

(3.22)

**Step 2: The Convergence of the \( \psi \)-series for Case (II)\)**

Similar to Case (I), by substituting the above equation into the system and rearranging terms, the system can be written in the form

\[ \ddot{f}_a + A_\alpha \dot{f}_a = \ddot{F}_a \]  

(3.23)

and the solution of the above first-order nonhomegenous vector ODE, for \( \alpha \geq 6 \), is

\[ \ddot{f}_a(z) = \int_{-\infty}^{z} e^{A_\alpha(x)} \ddot{F}_a(x) dx . \]  

(3.24)

After diagonalizing the matrix \( A_\alpha \), the eigenvalues of \( A_\alpha \) and their corresponding eigenvectors are found. The eigenvalues are

\[ \xi_1 = \alpha, \quad \xi_2 = \alpha + 1, \quad \xi_3 = \alpha - 6, \quad \xi_4 = \alpha - \sqrt{1 - 48\lambda} \]

Note that the eigenvalues are precisely \( \alpha - r_i \), where \( r_i \) are the resonances. Hence \( A_\alpha \) is diagonalizable as

\[ A_\alpha = P D_\alpha P^{-1} , \]  

(3.25)

where

\[
\begin{align*}
P &= \begin{vmatrix}
1 + \sqrt{1 - 48\lambda} & 1 - \sqrt{1 - 48\lambda} & 0 & 0 \\
24\lambda & 24\lambda & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 4 & 3 \\
0 & 0 & 1 & 1 \\
\end{vmatrix}, \\
P^{-1} &= \begin{vmatrix}
\frac{12\lambda}{\sqrt{1 - 48\lambda}} & \frac{1}{2} - \frac{1}{2\sqrt{1 - 48\lambda}} & 0 & 0 \\
\frac{1}{\sqrt{1 - 48\lambda}} & \frac{2}{2\sqrt{1 - 48\lambda}} & 0 & 0 \\
\frac{-12\lambda}{\sqrt{1 - 48\lambda}} & \frac{1}{2} + \frac{1}{2\sqrt{1 - 48\lambda}} & 0 & 0 \\
0 & 0 & 12 - 4 & 4 \\
0 & 0 & 12 & 7 \\
0 & 0 & 3 & 7 \\
\end{vmatrix}.
\end{align*}
\]
\[ D_\alpha = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha + 1 & 0 & 0 \\ 0 & 0 & \alpha - 6 & 0 \\ 0 & 0 & 0 & \alpha - \sqrt{1 - 48\lambda} \end{bmatrix}, \]

so

\[ \| \tilde{f}_\alpha(z) \| \leq \int_{-\infty}^{\infty} \| P \| \| e^{D_\alpha(xz)} \| \| P^{-1} \| \| \tilde{F}_\alpha(x) \| \, dx. \]

Let \( \| P \| \| P^{-1} \| = C \), where \( C \) is a constant depending on the value of \( \lambda \), and

\[ \| e^{D_\alpha(xz)} \| = e^{(\alpha-6)(xz)} , \]

so that

\[ \| \tilde{f}_\alpha(z) \| \leq C \int_{-\infty}^{\infty} e^{(\alpha-6)(xz)} |F_\alpha(x)| \, dx. \quad (3.26) \]

We now state and prove a theorem regarding the convergence of the \( \psi \)-series.

**Theorem 3.2.** There exists a positive number \( R \) such that the \( \psi \)-series converges \( \forall \tau, 0 < \tau < R \).

**Proof:**

We begin with the following

Claim: There exists a positive number \( K \) such that

\[ \| f_\alpha(z) \| \leq \frac{(2K+K\varphi)^\alpha}{\sqrt{\alpha - 1}} \]

where \( \varphi = e^{(\nu - \frac{1}{n})z} \)

**Proof of claim:**

From the above, we have

\[ f_\alpha(z) = \sum_{m=0}^{\alpha} (a_m^{(\alpha)} \cos(\mu_m z) + b_m^{(\alpha)} \sin(\mu_m z)) \, e^{m \frac{z}{n} (\nu - 1)} \]

\[ g_\alpha(z) = \sum_{m=0}^{\alpha} (c_m^{(\alpha)} \cos(\mu_m z) + d_m^{(\alpha)} \sin(\mu_m z)) \, e^{m \frac{z}{n} (\nu - 1)} \]
Since $|\cos(\mu \frac{m}{n} z)| \leq 1$, $|\sin(\mu \frac{m}{n} z)| \leq 1$, we have

$$|f_\alpha(z)| \leq \sum_{m=0}^\alpha |C_m^{(\alpha)}| e^{\frac{m}{n} (nv-1)z},$$

$$|g_\alpha(z)| \leq \sum_{m=0}^\alpha |D_m^{(\alpha)}| e^{\frac{m}{n} (nv-1)z},$$

where

$$C_m^{(\alpha)} = a_m^\alpha + b_m^\alpha,$$

$$D_m^{(\alpha)} = c_m^\alpha + d_m^\alpha.$$

The superscripts $(\alpha)$ simply indicate the dependence upon the $f_\alpha$ and $g_\alpha$; $C_m$ and $D_m$ are constants.

Since $k_\alpha(z)$ and $h_\alpha(z)$ are linear combinations of $f_\alpha(z)$ and $g_\alpha(z)$ and their derivatives, both $|k_\alpha(z)|$ and $|h_\alpha(z)|$ are dominated in the same manner as $|f_\alpha(z)|$ and $|g_\alpha(z)|$. (For definitions of $k_\alpha(z)$ and $h_\alpha(z)$, please refer to Chapter 2.)

By Lemma 2.1, with $M = 1$, $n_\alpha = \alpha$, $p > 0$, and given $N \geq 6$, there exist four positive constants, $k_f$, $k_g$, $k_h$, $k_k$, such that for $1 \leq \alpha \leq N-1$,

$$|f_\alpha(z)| \leq \frac{(pk_f+k_g \varphi)^\alpha}{\sqrt{\alpha + 1}},$$

$$|g_\alpha(z)| \leq \frac{(pk_g+k_h \varphi)^\alpha}{\sqrt{\alpha + 1}},$$

$$|h_\alpha(z)| \leq \frac{(pk_h+k_f \varphi)^\alpha}{\sqrt{\alpha + 1}},$$

$$|k_\alpha(z)| \leq \frac{(pk_k+k_g \varphi)^\alpha}{\sqrt{\alpha + 1}},$$

where $\varphi = e^{(v-1)\frac{z}{n}}$.

Therefore, by definition,

$$||f_\alpha(z)|| \leq \frac{(pk+k \varphi)^\alpha}{\sqrt{\alpha + 1}}, 0 \leq \alpha \leq N-1,$$

where $k = \max (k_f, k_g, k_h, k_k)$.

Let this be our induction hypothesis and let us prove that with the same $K$, the result is true for $\alpha = N$. First, we need to estimate $|F_\alpha(z)|$ for $N \geq 6$. 

3-14
For $N \geq 6$, we have
\begin{align*}
|F_N(z)| &\leq Af_N \cdot 2D \sum_{\beta=0}^{N-1} g_\beta f_{\alpha-\beta} \\
&\leq \frac{A}{\sqrt{N-1}} (pK + K\varphi)^{\frac{N-2}{2}} + 2D (pK + K\varphi)^N \sum_{\beta=1}^{N-1} \frac{1}{\sqrt{\beta + 1} \sqrt{N - \beta + 1}} \\
&\leq \left[ \frac{A}{\sqrt{N-1}} + 2D\pi \right] (pK + K\varphi)^N , \quad (3.29)
\end{align*}

where $\varphi = e^{(\nu \cdot \frac{1}{n})^2}$.

Substitution of the latter into (3.26) for $\alpha = N$ gives
\begin{align*}
\|f_N(z)\| &\leq C \left[ \frac{A}{\sqrt{N-1}} + 2D\pi \right] \int_{-\infty}^{\infty} e^{(\nu - \delta)(x - z)} \varphi \varphi (pK + K\varphi)^N \ dx \quad (3.30)
\end{align*}

(again, $\varphi = e^{(\nu \cdot \frac{1}{n})^2}$).

In order to get an estimate of the integral, we need to state and prove a lemma.

**Lemma 3.2.** Let $\alpha > \nu > \frac{1}{n}$, $a > 0$, $b > 0$, $\frac{b}{a} < \frac{1}{2(\nu - \frac{1}{n})}$, $z < 0$, $\alpha$ and $n$ are integers, and consider the integral
\begin{align*}
I(z) = \int_{-\infty}^{\infty} (a + be^{(\nu \cdot \frac{1}{n})^2})^\alpha e^{(\alpha - \delta)(x - z)} \ dx . \quad (3.31)
\end{align*}

Then
\begin{align*}
I(z) &\leq \frac{(a + b\varphi)^\alpha}{(\alpha - \delta)(1 - 2(\nu - \frac{1}{n})\xi)} , \quad \text{where } \xi = \frac{b}{a} . \quad (3.32)
\end{align*}

The proof is very similar to that of Lemma 4.3.3 in [16], with some minor modifications.

Now with $a = pK$, $b = K$, $\alpha = N$, $\delta = 6$ and $p = 2$, $n = 3$, $\nu = \frac{1}{2}$, we apply the above lemma to the integral in (3.30)
\begin{align*}
I(z) = \int_{-\infty}^{\infty} (pK + Ke^{(\nu \cdot \frac{1}{n})^2})^\alpha e^{(\alpha - 6)(x - z)} \ dx , \quad (3.33)
\end{align*}
to obtain

\[ I(z) \leq \frac{2(2K + K\varphi)^\alpha}{(N - 6)(1 - 2(\nu - \frac{1}{3}))}, \]

\[ \varphi = e^{(\nu - \frac{1}{\pi})z}. \]

We substitute into \( \| \hat{f}_N(z) \| \) to obtain

\[ \| \hat{f}_N(z) \| \leq 2C \left[ \frac{A}{\sqrt{N - 1}} + 2D\pi \right] \left( \frac{(2K + K\varphi)^\alpha}{(N - 6)(1 - 2(\nu - \frac{1}{3}))} \right), \]

\[ = 3C \left[ \frac{A}{\sqrt{N - 1}} + 2D\pi \right] \frac{(2K + K\varphi)^\alpha}{(N - 6)}, \quad (3.34) \]

\[ \varphi = e^{(\nu - \frac{1}{\pi})z}. \]

Hence the claim holds for \( \alpha = N \) if and only if \( N \) satisfies

\[ 3C \left[ \frac{A}{\sqrt{N - 1}} + 2D\pi \right] \frac{\sqrt{N + 1}}{N - 6} \leq 1 \]

This is our induction condition and is satisfied for sufficiently large \( N \), and the claim is proven.

Now we return to the proof of the theorem.

Since for all \( \alpha \geq 1 \), we have

\[ |f_\alpha(z)| \leq \| \hat{f}_\alpha(z) \|, \]

\[ \sum_{\alpha=0}^{\infty} |f_\alpha(z)|^\alpha \tau^{\alpha-2} \leq \sum_{\alpha=0}^{\infty} \frac{(2K + K\varphi)^\alpha}{\sqrt{\alpha + 1}} \tau^{\alpha-2}. \quad (3.35) \]

By the ratio test, the series is convergent if and only if

\[ (2K + K\varphi) \tau < 1; \quad \text{i.e.,} \quad 2K + K\varphi < e^\tau. \quad (3.36) \]

This is the condition for the existence of a positive radius of convergence. Equality holds in (3.36) for a unique negative \( z_0 \) such that for all \( z < z_0 \), (3.36) is true, which implies that the series in (3.35) converges for \( 0 < \tau < e^z \). Therefore the solution converges absolutely
for $0 < \tau < R$, where the radius of convergence is at least $e^\tau$. This completes the proof of the theorem.
Chapter 4

Complex Time

We have proved that the psi-series solutions of the H-H system are convergent on intervals of the form $0 < \tau < R$, where $\tau$ is real. If $\tau$ is permitted to be complex, then a branch cut must be made in order to render $\log \tau$ single-valued. Thus, if we let $\eta$ be real and consider the domain

$$D_\eta = \{ \tau \in \mathbb{C}: \tau = \rho e^{i\theta}, \rho > 0, \quad \eta \leq \theta < \eta + 2\pi \},$$

we shall see that the above convergence proofs are still valid if $\tau > 0$ is replaced by $\tau \in D_\eta$.

(1) In the case of an incompatible positive resonance, the phi-series solution can be written in the following format (see equations 2.5 and 2.6):

$$x(t) = \sum_{a=0}^{\infty} \left( \sum_{n=0}^{\infty} a_{a-m} \log^n \tau \right) \tau^{a-p}, \quad (4.1)$$

$$y(t) = \sum_{a=0}^{\infty} \left( \sum_{n=0}^{\infty} b_{a-m} \log^n \tau \right) \tau^{a-p}, \quad (4.2)$$

where $\gamma$ is an incompatible positive resonance.

Now if $\tau \in D_\eta$, then

$$x(t) = \sum_{a=0}^{\infty} \left( \sum_{n=0}^{\infty} c_n \log^n |\tau| \right) \tau^{a-p},$$

$$y(t) = \sum_{a=0}^{\infty} \left( \sum_{n=0}^{\infty} d_n \log^n \tau \right) \tau^{a-p},$$

where

$$c_n = \sum_{a=0}^{\infty} \sum_{n=0}^{\infty} a_{a-m} \left( \binom{l}{n} (i\theta)^l \right)^{a-n},$$

$$d_n = \sum_{a=0}^{\infty} \sum_{n=0}^{\infty} b_{a-m} \left( \binom{l}{n} (i\theta)^l \right)^{a-n}.$$
Thus the form of the solution is unaltered, except that \( z = \log \tau \) is replaced by \( z = \log |\tau| \) and the arbitrary constants \( a_{\tau,0}, b_{\tau,0} \) are replaced by the arbitrary constant \( c_0^{(n)}, d_0^{(n)} \). The system which governs \( f_a(z) \) for \( \tau > 0 \) also governs \( f_a(x, y) \) and \( g_a(x, y) \) for \( \tau \in D_n \).

Furthermore if we denote the solutions in (4.1) and (4.2) by \( G(\tau, \log \tau, a_{\tau,0}), H(\tau, \log \tau, b_{\tau,0}) \), then the solution for complex time is simply \( G(\tau, \log |\tau|, c^{(n)}_0), H(\tau, \log |\tau|, d^{(n)}_0) \). So the convergence proof for real \( \tau \) is also valid for complex time.

Note that in the \( \tau > 0 \) case, increasing \( |a_{\tau,0}| \) and \( |b_{\tau,0}| \) has the effect of increasing the value of the constant \( K \) in Lemma 2.1, thus decreasing the radius of convergence. In view of the relation between \( c_0^{(n)}, d_0^{(n)} \) and \( a_{\tau,0}, b_{\tau,0} \),

\[
c_0^{(n)} = a_{\tau,0} + a_{\tau,0} i \theta \quad (\theta \in (\nu, \nu + 2\pi)),
\]

\[
d_0^{(n)} = b_{\tau,0} + b_{\tau,0} i \theta \quad (\theta \in (\nu, \nu + 2\pi)),
\]

if \( a_{\tau,0}, b_{\tau,0} \) is fixed, but the order of the branch of the logarithm is increased, then for sufficiently large \( \nu \), \( c_0^{(n)} \) and \( d_0^{(n)} \) is increased. This suggests that going to higher branches of the solution has an adverse effect upon the radius of convergence. This result seems to be a confirmation of the “singularity clustering” and “self-similar” behaviour observed by other investigators [25], [26], [11].

(2) In the case of complex or irrational conjugate resonances, similar to (1), if the H-H system with leading order \( p \) and complex-conjugate resonance at \( l = \nu \pm i\mu \) has a solution for \( \tau > 0 \) denoted by \( G(\tau, \log \tau, a'_{\nu,0}, a'_{\nu,1}), H(\tau, \log \tau, b'_{\nu,0} b'_{\nu,1}) \), then the solution for the complex time is given by

\[
u(t) = G(\tau, \log |\tau|, \n [a'_{\nu,0} \cos \theta (\nu + p) - a'_{\nu,1} \sin \theta (\nu + p)] e^{-\mu t},
\]

\[
[a'_{\nu,0} \cos \theta (\nu + p) + a'_{\nu,1} \sin \theta (\nu + p)] e^{\mu t},
\]

\[
u(t) = G(\tau, \log |\tau|, \n [b'_{\nu,0} \cos \theta (\nu + p) - b'_{\nu,1} \sin \theta (\nu + p)] e^{-\mu t},
\]

\[
[b'_{\nu,0} \cos \theta (\nu + p) + b'_{\nu,1} \sin \theta (\nu + p)] e^{\mu t}.
\]

An analogous formula holds in the case of two irrational resonances.

In all these cases, the constant-coefficient equation governing \( f_a \) and \( g_a \) are unchanged when \( \tau > 0 \) is replaced by \( \tau \in D_n \), and therefore the convergence proof given for \( \tau > 0 \) is also valid for \( \tau \in D_n \).
Conclusion

In this thesis, we have dealt with series solutions of Hénon-Heiles systems, and we have seen that for certain values of their respective parameters, the general solutions of the Hénon-Heiles systems can be expressed in terms of series expansions about arbitrary ordinary points. In all other parameter regimes, the general solution contains movable poles and movable branch points (psi-series).

We have shown the convergence of the different types of the psi-series of the H-H system. Both real and complex time have been considered.

By doing some singularity analysis, Weiss et al. [11], [12], [13] have found that for some of the nonintegrable regimes in the H-H system, the structure of the natural boundaries on the same Reimann sheet have a self-similar structure. Furthermore, these singularity analyses have shed some light on non-integrable dynamical systems by offering useful (perhaps even convergent!) series expansion for these solutions near any one of the singularities of the system. The results in this thesis are confirmations of the above.

In this thesis, we did not consider the case where the system has a negative resonance (other than the usual -1). Conte et al. have discussed negative resonances in [14]. In their paper, they modify the Painlevé test so that negative resonance can be treated.

Another interesting issue is the relationship between weak Painlevé and Painlevé. It may well turn out that all integrable weak Painlevé equations can be transformed to full Painlevé ones through some ingenious, highly nontrivial transformations of both dependent and independent variables. It is clear that more study is needed in this direction.
## Appendix

Table 1: Symbols and their representation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>the parameters of the H-H system</td>
</tr>
<tr>
<td>$\tau$</td>
<td>$t - t_0$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>resonance</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>leading order</td>
</tr>
<tr>
<td>$\xi$</td>
<td>eigenvalue</td>
</tr>
<tr>
<td>$p$</td>
<td>arbitrary</td>
</tr>
<tr>
<td>$l, k, i$</td>
<td>index</td>
</tr>
<tr>
<td>$q$</td>
<td>arbitrary</td>
</tr>
<tr>
<td>$\nu, \mu$</td>
<td>the real and complex part for resonance</td>
</tr>
</tbody>
</table>
Bibliography


7-2


