NOTICE

The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us a poor photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED

AVIS

La qualité de cette microfiché dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de mauvaise qualité.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formules d'autorisation qui accompagnent cette thèse.

LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS RÉCU
PERMISSION TO MICROFILM — AUTORISATION DE MICROFILMER

- Please print or type — Écrire en lettres moulées ou dactylographier

<table>
<thead>
<tr>
<th>Full Name of Author — Nom complet de l'auteur</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SUZANNE MARIE SEAGER</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Date of Birth — Date de naissance</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>22 MAY 1959</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Country of Birth — Lieu de naissance</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CANADA</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Permanent Address — Résidence fixe</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>200 COW DRIVE UNIT 97</strong></td>
</tr>
<tr>
<td><strong>OTTAWA, ONTARIO</strong></td>
</tr>
<tr>
<td><strong>CANADA K1V 9P7</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Title of Thesis — Titre de la thèse</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A Bound on the Rank of Solvable Primitive Permutation Groups</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>University — Université</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CARLETON UNIVERSITY</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Degree for which thesis was presented — Grade pour lequel cette thèse fut présentée</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>P.H.D.</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year this degree conferred — Année d'obtention de ce grade</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1985</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Name of Supervisor — Nom du directeur de thèse</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>DR. JOHN DIXON</strong></td>
</tr>
</tbody>
</table>

Permission is hereby granted to the NATIONAL LIBRARY OF CANADA to microfilm this thesis and to lend or sell copies of the film.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

L'autorisation est, par la présente, accordée à la BIBLIOTHEQUE NATIONALE DU CANADA de microfomer cette thèse et de prêter ou de vendre des exemplaires du film.

L'auteur se réserve les autres droits de publication; ni la thèse ni de longs extraits de celle-ci ne doivent être imprimés ou autrement reproduits sans l'autorisation écrite de l'auteur.

<table>
<thead>
<tr>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>25 JANUARY 1985</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Signature]</td>
</tr>
</tbody>
</table>
A Bound on the Rank of
Solvable Primitive Permutation Groups

by

Suzanne Seager

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics

Carleton University
Ottawa, Ontario
November 30, 1984

© copyright
1984, Suzanne Seager
The undersigned hereby recommend to
The Faculty of Graduate Studies and Research
acceptance of the thesis, submitted by
SUZANNE SEAGER
in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

Chairman, Department of Mathematics

Thesis Supervisor

External Examiner

Carleton University

January 25, 1985
Abstract

A Bound on the Rank of
Solvable Primitive Permutation Groups

Suzanne Seager

Theorem Let $G$ be a maximal solvable primitive permutation group of degree $d$ and rank $r$. Then either

$$r > 24^{1/3}d^{0.92747-1.5}$$

or $G$ is permutation isomorphic to the wreath product

$S(p^m) \wr K$, where $S(p^m)$ is the group of all semilinear $G$-transformations on $GF(p^m)$, $K$ is a maximal solvable transitive subgroup of $S_k$, $d = p^{mk}$, and

$$r > 24^{1/3}(1.188)^k-1.5$$

This theorem is proved by considering separately the cases that the stabilizer of $G$ is primitive or imprimitive as a linear group and constructing bounds in each of these two cases. A class of groups acting as a limiting case is found. This extends work of Foulser and Dornhoff.
Acknowledgements

I would like to express my gratitude to my supervisor Dr. John Dixon, to Dr. John Poland for his assistance, and to my family for their support.
# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>iii</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>iv</td>
</tr>
<tr>
<td>List of Tables</td>
<td>vi</td>
</tr>
<tr>
<td>List of Symbols</td>
<td>vii</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Chapter I. Primitive Solvable Permutation Groups</td>
<td>3</td>
</tr>
<tr>
<td>Chapter II. The Primitive Case</td>
<td></td>
</tr>
<tr>
<td>2.1 Maximal Solvable Primitive Linear Groups</td>
<td>5</td>
</tr>
<tr>
<td>2.2 A Bound on the Number of Orbits</td>
<td>7</td>
</tr>
<tr>
<td>2.3 The Case ( n = 2m )</td>
<td>9</td>
</tr>
<tr>
<td>2.4 The Number of Orbits for ( n = 2m )</td>
<td>12</td>
</tr>
<tr>
<td>2.5 A General Bound</td>
<td>18</td>
</tr>
<tr>
<td>Chapter III. The Imprimitive Case</td>
<td></td>
</tr>
<tr>
<td>3.1 The Wreath Product</td>
<td>23</td>
</tr>
<tr>
<td>3.2 The Number of Orbits of a Wreath Product</td>
<td>27</td>
</tr>
<tr>
<td>3.3 Some Small Cases</td>
<td>29</td>
</tr>
<tr>
<td>3.4 The Case ( K ) is Primitive</td>
<td>35</td>
</tr>
<tr>
<td>3.5 A Bound on ( r(k,h) )</td>
<td>38</td>
</tr>
<tr>
<td>3.6 A Limiting Case</td>
<td>42</td>
</tr>
<tr>
<td>3.7 An Exact Formula for ( r(k,h) )</td>
<td>45</td>
</tr>
<tr>
<td>Chapter IV. The General Bound</td>
<td></td>
</tr>
<tr>
<td>4.1 The Product Action of the Wreath Product</td>
<td>56</td>
</tr>
<tr>
<td>4.2 Degrees with Small Rank</td>
<td>58</td>
</tr>
<tr>
<td>4.3 The Main Theorem</td>
<td>60</td>
</tr>
<tr>
<td>References</td>
<td>65'</td>
</tr>
</tbody>
</table>
List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table I</td>
<td>Formulas for $r^*(p,n,n/2)$</td>
<td>16</td>
</tr>
<tr>
<td>Table II</td>
<td>Values of $r^*(p,n,n/2)$</td>
<td>17</td>
</tr>
<tr>
<td>Table III</td>
<td>Values of $r^*(p^n)$ less than 1000</td>
<td>22</td>
</tr>
<tr>
<td>Table IV</td>
<td>Formulas for $(h)R_\lambda$</td>
<td>33</td>
</tr>
<tr>
<td>Table V</td>
<td>Values of $(h)R_e$ at most 1000</td>
<td>34</td>
</tr>
<tr>
<td>Table VI</td>
<td>Values of $a_h$ and $a'_h$</td>
<td>44</td>
</tr>
<tr>
<td>Table VII</td>
<td>Values of $r(k,h)$ at most 1000</td>
<td>55</td>
</tr>
<tr>
<td>Table VIII</td>
<td>Semilinear-Component Groups</td>
<td>58</td>
</tr>
<tr>
<td>Table IX</td>
<td>Non-Semilinear-Component Groups</td>
<td>59</td>
</tr>
</tbody>
</table>
List of Symbols

\([x,y]\) is the commutator \(x^{-1} y^{-1} xy\) of \(x\) and \(y\).

\((m,n)\) is the greatest common divisor of \(m\) and \(n\).

\((a/p)\) is the Legendre symbol; \((a/p) = 1\) if and only if \(a\) is a quadratic residue of \(p\).

\(\text{GF}(p^n)\) is the Galois field with \(p^n\) elements.

\(\text{Gal}(K/F)\) is the Galois group of field \(K\) over field \(F\).

\(\text{Sym}(X)\) is the symmetric group on the set \(X\).

\(S_n\) is the symmetric group on \(n\) elements.

\(\text{GL}(n,p)\) is the general linear group of degree \(n\) over \(\text{GF}(p)\).

\(\text{SL}(n,p)\) is the special linear group of degree \(n\) over \(\text{GF}(p)\).

\(\text{Sp}(2\ell,p)\) is the symplectic group of degree \(2\ell\) over \(\text{GF}(q)\).

\(C_G(H)\) is the centralizer of the group \(H\) in the group \(G\).

\(|G|\) is the order of the group \(G\).

\(G|_U\) is the restriction of the group \(G\) to the set \(U\).

\(o(g)\) is the order of the group element \(g\).

\(\text{Fix}(g)\) is the set of all fixed points of the element \(g\).

Symbols Defined in the Text

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T(p^n))</td>
<td>3</td>
</tr>
<tr>
<td>(S(p^n))</td>
<td>4</td>
</tr>
<tr>
<td>(r^*(p,n,m))</td>
<td>8</td>
</tr>
<tr>
<td>(r^*(p^n))</td>
<td>18</td>
</tr>
<tr>
<td>((h)K)</td>
<td>29</td>
</tr>
<tr>
<td>(r(k,h))</td>
<td>38</td>
</tr>
<tr>
<td>(H \wr K)</td>
<td>23, 25, 56</td>
</tr>
</tbody>
</table>
Introduction

In 1957 Huppert [5] proved that with six exceptions, any solvable 2-transitive permutation group is semilinear. This was extended in one direction by Passman [11], who found the exceptional cases for solvable 3/2-transitive groups. In another direction Foulser [4] and Dornhoff [5] found the exceptions for primitive solvable permutation groups of rank 3, and in addition Foulser considered the possible exceptions for rank 4. Dornhoff also showed that for any primitive non-semilinear solvable permutation group $\mathcal{G}$ such that the stabilizer of $\mathcal{G}$ is a primitive linear group, there exists some function of the rank of $\mathcal{G}$ which bounds the degree. In this thesis we shall extend this to find a relationship between the degree and the rank of a primitive solvable permutation group.

In Chapter I we give the standard analysis of a primitive solvable permutation group $\mathcal{G}$, by which the stabilizer $G$ of $\mathcal{G}$ is found to be equivalent to an irreducible linear group. This may be semilinear, in which case $\mathcal{G}$ will have rank 2 whatever its degree, but otherwise some restriction can be found on the rank of $\mathcal{G}$, and thus in the next two chapters we consider irreducible linear groups which are not semilinear.

In Chapter II we consider the case that $G$ is a solvable primitive linear group. We use the results of Suprunenko [14]
to analyze the structure of $G$, and we find a lower bound on the number of orbits of $G$ given the degree, and a class of groups which act as a limiting case.

In Chapter III we consider the case that $G$ is a solvable imprimitive linear group. Here we establish a lower bound on the number of orbits of $G$ based on the degrees and numbers of orbits of its primitive components.

Finally in Chapter IV we combine the results of each of these cases to give a bound on the rank of $G$.
Chapter I. Primitive Solvable Permutation Groups

The standard analysis of primitive solvable permutation groups follows (see, for example, [12] or [4]):

Theorem 1.1 Let $\mathcal{G}$ be a primitive solvable subgroup of degree $d$, and let $V$ be a minimal normal subgroup of $\mathcal{G}$. Then $V$ is elementary abelian of order $p^n$, for some prime $p$ and $n \geq 1$, and is a regular normal subgroup of $\mathcal{G}$ (so $d = p^n$). Let $G = \mathcal{G}_0$ be the stabilizer of $\mathcal{G}$. Then $G$ acts as an irreducible subgroup of $GL(V)$.

The converse is also true ([1]):

Theorem 1.2 Let $G$ be an irreducible solvable subgroup of $GL(V)$. Then the split extension $\mathcal{G} = V \cdot G$ is a primitive solvable permutation group acting on $V$.

So now let $\mathcal{G}$ be a primitive solvable permutation group of degree $d = p^n$, as in Theorem 1.1, with stabilizer $G$ an irreducible solvable subgroup of $GL(n,p)$ ($= GL(V)$). Then the rank of $\mathcal{G}$ is equal to the number of orbits of $G$. Since we wish to minimize rank for a fixed degree, we may assume that $\mathcal{G}$ is a maximal solvable subgroup of the symmetric group $S_d$, and thus that $G$ is a maximal irreducible solvable subgroup of $GL(n,p)$.

Suppose first that $V$ can be identified with the field $K = GF(p^n)$ so that $G = T(p^n)$, the group of all semilinear
transformations of $V$ of the form $x + ax^\sigma$, with $a \in K$, $a \neq 0$, and $\sigma$ a field automorphism in $\text{Gal}(K)$. Then $G = S(p^n)$, the semilinear group of all transformations of $V$ of the form $x + ax^\sigma + b$, with $a, b \in K$, $a \neq 0$, and $\sigma$ in $\text{Gal}(K)$. $S(p^n)$ is solvable of order $np^n(p^n - 1)$ and rank 2 (see [12] Chapter 19).

So we will now be concerned with the maximal solvable irreducible subgroups of $\text{GL}(n, p)$ which are not semilinear. By Theorem 1.2, finding the possible numbers of orbits of such groups will determine the possible ranks of maximal solvable primitive permutation groups which are not semilinear.
Chapter II. The Primitive Case

1. Maximal Solvable Primitive Linear Groups

The following material is from Suprunenko [14], Chapter 20.

Let $G$ be a maximal primitive solvable subgroup of $GL(n,p)$. Let $F$ be the maximal abelian normal subgroup of $G$, $V$ the centralizer of $F$ in $G$, and $A/F$ the maximal abelian normal subgroup of $G/F$ contained in $V/F$. The invariant series $G \supseteq V \supseteq A \supseteq F \supseteq 1$ is unique, and is called the Suprunenko series of $G$. We shall now study the factors of this series more closely.

Let $P = GF(p)$. Then $F = K^*$, for some extension field $K$ of $P$. Let $m$ be the index of $F$ in $K$; then $m | n$, and $m = n$ if and only if $G$ is semilinear. Further $G/V$ is isomorphic to a subgroup of $Gal(K/P)$, and $V$ is an absolutely irreducible subgroup of $GL(n/m, K)$.

Let $n/m = q_1^{l_1} \cdots q_k^{l_k}$ be the canonical decomposition of $n/m$, and let $Q_j/F$ be a Sylow $q_j$-subgroup of $A/F$, $j=1, \ldots, k$. Then the $Q_j$ commute among themselves elementwise. For each $j$, $Q_j$ can be represented in the form

$$Q_j = \langle u_{1j} \rangle \langle v_{1j} \rangle \cdots \langle u_{kj} \rangle \langle v_{kj} \rangle^F$$

where $[u_{ij}, v_{ij}] = \eta$ for some $\eta$ in $F$ such that $\eta^q_j = 1$, $(u_{1j})^{q_j}, (v_{1j})^{q_j} \in F$, and elements in distinct pairs commute.
Now for each $x$ in $V$ suppose
\[ xu_x^{-1} = \lambda_i (u_1)^{\alpha_{ij}} \ldots (u_{\ell_j})^{\alpha_{ij}} (v_1)^{\gamma_{ij}} \ldots (v_{\ell_j})^{\gamma_{ij}} \]
and
\[ xv_x^{-1} = \mu_i (u_1)^{\beta_{ij}} \ldots (u_{\ell_j})^{\beta_{ij}} (v_1)^{\delta_{ij}} \ldots (v_{\ell_j})^{\delta_{ij}} \]
for $\lambda_i, \mu_i$ in $K^*, \ 0 \leq \alpha_{ri}, \beta_{ri}, \gamma_{ri}, \delta_{ri} < q_j, \ i = 1, \ldots, \ell_j$.

Then we define a homomorphism $\psi_j : V \to \text{Sp}(2\ell_j, q_j)$ as follows:

\[
\psi_j(x) = \begin{pmatrix}
\alpha_{11} & \ldots & \alpha_{1\ell_j} & \beta_{11} & \ldots & \beta_{1\ell_j} \\
\vdots & & \vdots & \vdots & & \vdots \\
\alpha_{\ell_j1} & \ldots & \alpha_{\ell_j\ell_j} & \beta_{\ell_j1} & \ldots & \beta_{\ell_j\ell_j} \\
\gamma_{11} & \ldots & \gamma_{1\ell_j} & \delta_{11} & \ldots & \delta_{1\ell_j} \\
\vdots & & \vdots & \vdots & & \vdots \\
\gamma_{\ell_j1} & \ldots & \gamma_{\ell_j\ell_j} & \delta_{\ell_j1} & \ldots & \delta_{\ell_j\ell_j}
\end{pmatrix}
\]

Note that since $A$ is a normal subgroup of $G$ we could as easily have defined $\psi_j : G \to \text{GL}(2\ell_j, q_j)$. Then the mapping

\[
\overline{\psi} : V/A \to \prod_{j=1}^{k} \text{Sp}(2\ell_j, q_j)
\]

induced on $V/A$ by the $\psi_j$ is a monomorphism. Thus $V/A$ is isomorphic to a solvable subgroup of the direct product of the $k$ symplectic groups $\text{Sp}(2\ell_j, q_j), j = 1, \ldots, k$.

Thus, based on its Suprunenko series, we have the following structure for $G$:
2. A Bound on the Number of Orbits

Suppose $G$ has the structure given above. Let $s_1(q^\ell_j)$ be the maximal order of a solvable subgroup of $\text{Sp}(2\ell_j, q_j)$ and let $s_1(n/m) = \prod_{j=1}^{k} s_1(q^\ell_j)$. Then $|\psi_j(V)| \leq s_1(q^\ell_j)$ and so $|V/A| \leq s_1(n/m)$. Since $|G| = |G/V| |V/A| |A/F| |F|$, 

$|G| \leq ms_1(n/m) (n/m)^2 (p^m - 1)$

Now consider the action of $G$ on $Q_j/F$ by conjugation. This action is completely reducible, since for any normal abelian subgroup $B/F$ of $Q_j/F$, $Q_j/F = (B/F) \times (C_{Q_j}(B)/F)$ by Isaacs [7] Theorem 2. Thus if we extend the definition of $\psi_j$ to $\psi_j: G \to \text{GL}(2\ell_j, q_j)$ we have $\psi_j(G)$ completely reducible for $j=1, \ldots, k$. Let $R = \{x \in G \mid \psi_j(x) = 1, j=1, \ldots, k\}$. Then $A \leq R$, $V \cap R = A$, and so $|R/A| \leq |G/V| \leq m$, and we have the monomorphism $\bar{\psi}: G/R \to \prod_{j=1}^{k} \text{GL}(2\ell_j, q_j)$, with $\bar{\psi}(G/R) = \prod_{j=1}^{k} \psi_j(G)$. 

$G/V \leftrightarrow \text{Gal}(K/F)$

$V/A \leftrightarrow \prod_{j=1}^{k} \text{Sp}(2\ell_j, q_j)$, where $\prod_{j=1}^{k} q_j^{2\ell_j} = |A/F|$
Let $s_2(q_j^{l_j})$ be the maximal order of a solvable completely reducible subgroup of $GL(2q_j^{l_j}, q_j)$, and $s_2(n/m) = \prod_{j=1}^{k} s_2(q_j^{l_j})$. Then $|\psi_j(G)| \leq s_2(q_j^{l_j})$ so $|G/R| \leq s_2(n/m)$. Then since $|G| = |G/R| |R/A| |A/F| |F|$, we have

$$|G| \leq s_2(n/m) m(n/m)^2 (p^m-1).$$

Now let $s(n/m) = \min(s_1(n/m), s_2(n/m))$. Then we have

$$|G| \leq s(n/m) m(n/m)^2 (p^m-1) \quad (1)$$

But by Wolf [15] Theorem 3.1, $s_2(n/m) \leq (n/m)^{2\alpha (24^{1/3})}$, for $\alpha = (3\log 48 + \log 24)/3\log 9 = 2.24399...$, and so

$$|G| \leq m(n/m)^{2+2\alpha} (p^m-1)/(24^{k/3}) \quad (2)$$

Now define $r^*(p,n,m)$ to be the minimal possible number of nontrivial orbits of a maximal solvable subgroup $G$ of $GL(n,p)$ such that its maximal abelian normal subgroup has order $p^m-1$. Then $G$ acts on a space of $p^n-1$ elements, and so $r^*(p,n,m) \geq (p^n-1)/|G|$. Thus using (1) we have

$$r^*(p,n,m) \geq (p^n-1)/s(n/m) m(n/m)^2 (p^m-1) \quad (3)$$

or, using (2),

$$r^*(p,n,m) \geq 24^{k/3} (p^n-1)/m(n/m)^{2+2\alpha} (p^m-1) \quad (4)$$

and so we have a lower bound on the number of orbits.
3. **The Case** \( n = 2m \)

We now construct a specific class of maximal solvable primitive groups. We assume that the structure of \( G \) is as above with \( n = 2m \). The case \( m = 1 \) was developed in Suprunenko [14] Chapter 21, and that for \( m = 2 \) in Zhavrid [16]. The following repeats their constructions for arbitrary \( m \).

Since \( n/m = 2 \), we have \( A = \langle a \rangle \langle b \rangle F \), where \( a^2, b^2 \in F \) and \([a,b] = -1\), and \( F = K^* \) for \( K = GF(p^m) \). By Theorem 21.5 and Lemma 21.4 of Suprunenko, there are two possibilities for \( a \) and \( b \) as elements of \( GL(2,K) \):

(i) \( K \) contains an element \( i \) of order 4, and
\[
 a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{or}
\]

(ii) \( K \) contains no element of order 4, and
\[
 a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} \theta & \omega \\ \omega & -\theta \end{pmatrix}, \quad \text{where} \ \theta, \omega \in P, \\
\text{and} \ \theta^2 + \omega^2 = -1.
\]

But in the first case, we may replace \( a \) by \( a' = ia \), and \( b \) by \( b' = ib \) to get \( a' \) and \( b' \) such that \((a')^2 = (b')^2 = -1 \) and \([a',b'] = -1\), and so as in Theorem 21.5 of Suprunenko, we may instead have
\[
 a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} \theta & \omega \\ \omega & -\theta \end{pmatrix}, \quad \theta, \omega \in P, \ \theta^2 + \omega^2 = -1.
\]
Thus in either case we may assume
\[
a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} \theta & \omega \\ \omega & -\theta \end{pmatrix}, \quad \text{where} \quad \theta, \omega \in \mathbb{P}, \quad \theta^2 + \omega^2 = -1,
\]
and \(\omega = 0\) if and only if there is an element in \(\mathbb{P}\) or order 4. Further, if \(\rho\) be an element of order \(p^{m-1}\) in \(\mathbb{K}^*\), then \(\mathbb{K}\) is generated by the element \(c = \rho I\), and thus \(A = \langle a, b, c \rangle\).

Now we wish to find \(V\) such that \(\psi(V) = \text{Sp}(2,2)\). But \(\text{Sp}(2,2) = \text{GL}(2,2)\) and is generated by \(\tau_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) and \(\tau_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), and thus we want \(v_1, v_2\) such that \(\psi(v_i) = \tau_i, i = 1, 2\). But \(\psi(v) = \tau_1\) if and only if \(vav^{-1} = \lambda a\) and \(v bv^{-1} = \mu b\), for some \(\lambda, \mu \) in \(\mathbb{K}^*\), and thus \(\psi(v) = \tau_1\) if and only if \(v \in v_1 A\) for \(v_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\). Similarly \(\psi(v) = \tau_2\) if and only if \(v \in v_2 A\) for \(v_2 = \begin{pmatrix} \theta & \omega -1 \\ \omega +1 & -\theta \end{pmatrix}\). Thus we have \(V = \langle v_1, v_2, a, b, c \rangle\) and \(\psi(V) = \text{GL}(2,2)\).

Now we want to find \(G\) such that \(G/V \cong \text{Gal}(\mathbb{K}/\mathbb{P})\). Let \(\phi\) be the mapping \(x \rightarrow x^p\) for all \(x\) in \(\mathbb{K}\). Then \(\phi\) generates \(\text{Gal}(\mathbb{K}/\mathbb{P})\) and so we want to find \(h\) in \(\text{GL}(n,p)\) such that \(hAh^{-1} = A\) and \(huh^{-1} = u^\phi\) for all \(u\) in \(F\); i.e. such that \(\psi(h) \in \text{GL}(2,2)\) and \(hch^{-1} = c^\phi\). Now if \(\psi(h) \in \text{GL}(2,2)\) then since \(\psi(V) = \text{GL}(2,2)\) there exists some \(v\) in \(V\) with \(\psi(v) = \psi(h)\) and thus replacing \(h\) with \(hv^{-1}\) we may assume that \(\psi(h) = I\) and thus that \(hah^{-1} = \lambda a\) and \(h bh^{-1} = \mu b\) for some \(\lambda, \mu \) in \(\mathbb{K}^*\).
But this is true if and only if \( h \in gA \), where \( g = \begin{bmatrix} 1 & 0 \\ 0 & \Sigma \end{bmatrix} \)

for some \( \Sigma \) in \( GL(m, p) \) satisfying \( \Sigma v \Sigma^{-1} = v^0 \) for all \( v \) in \( K \).
Thus we have constructed \( G = <g, v_1, v_2, a, b, c> \), and we must now show that it is maximal solvable primitive.

Theorem 2.1 Let \( G = <g, v_1, v_2, a, b, c> \) as above. Then \( G \) is a maximal solvable primitive subgroup of \( GL(n, p) \), and any maximal solvable primitive subgroup of \( GL(n, p) \) satisfying \( n = 2m \) is conjugate in \( GL(n, p) \) to \( G \).

Proof: Let \( V = <v_1, v_2, a, b, c> \). Then \( V \) is normal in \( G \) and by Lemma 21.5 of Suprunenko, \( V \) is primitive as a subgroup of \( GL(2, K) \). But then \( V \) is primitive as a subgroup of \( GL(n, p) \).

For, (following the proof of Lemma 5 of Konyukh [8]), suppose \( L_1 + \ldots + L_t \) is a system of imprimitivity for \( V \) as a subgroup of \( GL(n, p) \). Let \( f \) be an element of \( F \), and let \( x_i \) be in \( L_i \) for some \( i \). Suppose \( f(x_i) = x_j \in L_j \). Then \((I + f)(x_i) = x_i + x_j \).

But \( I + f \in F \), and so \((I + f):(L_i + L_s) \) for some \( s \). Thus \( x_i + x_j \in L_s \), and so \( i = j = s \). Therefore \( f(L_i) = L_i \), and so \( L_i \) is \( F \)-invariant for \( i = 1, \ldots, t \), and so each \( L_i \) is a \( K \)-space.

But then \( L_1 + \ldots + L_t \) is a system of imprimitivity for \( V \) as a subgroup of \( GL(2, K) \), a contradiction. Thus \( G \) is primitive, and \( G \) is solvable since each of \( G/V, V/A, A/F \), and \( F \) is solvable. Finally, by the construction of \( G \), \( G \) is maximal and any maximal solvable primitive subgroup of \( GL(n, p) \) satisfying \( n = 2m \) is conjugate in \( GL(n, p) \) to \( G \).
4. The Number of Orbits for \( n = 2m \)

Let \( G = \langle g, v_1, v_2, a, b, c \rangle \) as above. Then we have
\[
g = \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} \theta & \omega - 1 \\ \omega + 1 & -\theta \end{bmatrix}, \quad a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\]
\[
b = \begin{bmatrix} \theta & \omega \\ \omega & -\theta \end{bmatrix}, \quad \text{and} \quad c = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}, \quad \text{where} \quad \langle \rho \rangle = K^* \text{ for } K = GF(p^m),
\]

\( \theta, \omega \in F \) such that \( \theta^2 + \omega^2 = -1 \), and \( I \in GL(m,p) \) such that \( \Sigma \rho^i \omega^{-1} = \rho^0 \), where \( o: x \rightarrow x^p \) generates \( Gal(K/F) \) (\( F = GF(p) \)).

Further \( V = \langle v_1, v_2, a, b, c \rangle \), \( A = \langle a, b, c \rangle \), and \( F = \langle c \rangle \). We consider first the conjugacy classes of \( G \).

Now \( g v g^{-1} = v \) for \( v = a, b, v_1, v_2 \), and thus \( gF \) centralizes \( V/F \) in \( G/F \). Therefore \( G/F \) is equivalent to \( \langle gF \rangle \times V/F \). But \( V/F \) has 24 elements comprising five conjugacy classes, each class \( C_i \) having length \( l_i \) and representative \( c_i \) of order \( o_i \) as follows:

<table>
<thead>
<tr>
<th>( C_i )</th>
<th>( l_i )</th>
<th>( o_i )</th>
<th>( c_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>4</td>
<td>( \begin{bmatrix} 1 &amp; -1 \ 1 &amp; 1 \end{bmatrix} )</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>2</td>
<td>( \begin{bmatrix} \theta &amp; \omega - 1 \ \omega + 1 &amp; -\theta \end{bmatrix} )</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>( \begin{bmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{bmatrix} )</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>3</td>
<td>( \begin{bmatrix} \theta - \omega - 1 &amp; \theta + \omega - 1 \ \theta + \omega + 1 &amp; -\theta - \omega - 1 \end{bmatrix} )</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>I</td>
</tr>
</tbody>
</table>

Conjugacy Classes of \( V/F \)
Thus $G/F$ has $5m$ conjugacy classes $C_{i1}, \ldots, C_{5m}$, where the class $C_{ij}$ has length $l_i$ and class representative $g^j c_i F$, for $i=1, \ldots, 5$ and $j=1, \ldots, m$. We can now count the number of orbits of $G$.

**Theorem 2.2** Let $G = \langle g, v_1, v_2, a, b, c \rangle$ as above, $m = n/2$. Let $\alpha_1 = -1$, $\alpha_2 = -2$, $\alpha_3 = -1$, $\alpha_4 = -3$, and $\alpha_5 = 1$. Define

$$
\phi^i_j = \begin{cases} 
  p^{(m,j)+1} & \text{if } \alpha_i \mid m/(m,j) \\
  1 & \text{else if } p = 3, i = 4 \\
  2 & \text{else if } 2 \mid (m,j) \text{ or } (\alpha_i/p) = 1 \\
  0 & \text{else} 
\end{cases}
$$

Then the number of nontrivial orbits of $G$ is

$$r^*(p,n,n/2) = \frac{n/2 \cdot 5}{(\Sigma_{j=1}^{m} l_j \phi^i_j)/12}$$

**Proof:** $G$ acts on the space $\mathbb{P}^{(n)}$ which can be identified with the space $K^{(2)}$. Since $F = \langle c \rangle$ is the maximal abelian normal subgroup of $G$, and $F$ acts on $K^{(2)}$ as multiplication by elements of $K$, the number of nontrivial orbits of $G$ on $K^{(2)}$ is equal to the number of orbits of $G/F$ on $K^{(2)}$, the projective space obtained from $K^{(2)}$. But, by Burnside's Lemma, the number of orbits of $G/F$ on $K^{(2)}$ is

$$\Sigma_{x \in G/F} \text{Fix}(x)/|G/F|$$

and so, using the conjugacy class structure of $G$, we have

$$r^*(p,n,n/2) = \frac{m \cdot 5}{(\Sigma_{j=1}^{m} l_j \text{Fix}(g^j c_i F))/24m}$$
Thus to prove the theorem it suffices to show that
\[ \text{Fix}(g_j^i c_1 F) = \phi_j^i \] for \( i=1,2, \ldots, 5 \), \( j=1,2, \ldots, m \).

Let \( i \neq 1 \) and consider the action of \( g_j^i c_1 F \) on the
projective space \( \mathbb{P}^2(2) = \{ (0^1) \} \cup \{ (1_v^1) \mid v \in K \} \) of \( p+1 \)
points. Since \( g_j^i c_1 = \begin{pmatrix} E^j & -s^j \\ E^j & s^j \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) is not a fixed point.

Suppose \( \begin{pmatrix} 1 \\ v \end{pmatrix} \) is a fixed point. Then \( g_j^i c_1 \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ v' \end{pmatrix} \) for
some \( v' \) in \( K \), and so \( (1 - v(\sigma^j))v = (1 + v(\sigma^j))v \), or
\[ v(\sigma^j) = \frac{v-1}{v+1}, \quad v \neq 1,-1 \] \hspace{1cm} (5)

Let \( t = \sigma(\sigma^j) = m/(m,j) \). Then \( v(\sigma^{tj}) = v \). But (5) implies
\[ v(\sigma^{4j}) = v \]. Thus we have
\[ v(\sigma^{4,4j}) = v \] \hspace{1cm} (6)

So we consider now three cases:

(i) \( 2 \nmid t \). Then (6) implies \( v(\sigma^{4j}) = v \), and so by (5), \( v^2 + 1 = 0 \). This has two solutions in \( K \) if \((-1/p) = 1 \) or \( 2 \mid (m,j) \) and none otherwise, since \( v \) must be in the field \( GF(p^{m,j}) \), the fixed field of \( \sigma^{m,j} \) in \( K \). Further any solution of \( x^2 + 1 = 0 \) in \( GF(p^{m,j}) \) generates a fixed point and thus the number of fixed points is two if \((-1/p) = 1 \) or \( 2 \mid (m,j) \) and none otherwise.

(ii) \( 2 \mid t \), \( 4 \nmid t \). Then (6) implies \( v(\sigma^{2j}) = v \), and so by (5), \( v = -1/v \), and as above the number of fixed points
is two if \((-1/p) = 1\) or \((m,j)\) and none otherwise.

(iii) \(4 \mid t\). Then \((4j,m) = 4(j,m)\), and so since
\[\nu(o^{4j}) = \nu \quad \text{and} \quad \nu(o^m) = \nu,\]
we have \(\nu(o^{4(j,m)}) = \nu\). But
\[j/(j,m) \equiv 1 \text{ or } 3 \pmod{4}.\]
Thus \(\nu(o^j) = \nu(o^{(j,m)})\) or \(\nu(o^j) = \nu(o^{3(j,m)})\). If \(\nu(o^j) = \nu(o^{(j,m)})\), then by (5)
\[\nu(o^{(j,m)}) + \nu(o^{(j,m)}) = \nu + 1 = 0,\]
so \(\nu\) is a solution of
\[x^{p^{(m,j)}+1} + x^{p^m} + x + 1 = 0\]
in the field \(K = \text{GF}(p^m)\). But any solution to this equation in any extension field
of \(p = \text{GF}(p)\) satisfies \(x^{p^{4(m,j)}} = x\), and thus satisfies \(x^{p^m} = x\) and is in \(K = \text{GF}(p^m)\). Thus the number of fixed
points is equal to the number of solutions of the equation
above, which is \(p^{(m,j)} + 1\). Similarly if \(\nu(o^j) = \nu(o^{3(j,m)})\)
then applying \(o^{3(j,m)}\) to (5) gives
\[\nu = \frac{\nu(o^{(j,m)})}{\nu(o^{(j,m)}) + 1}\]
and the same argument again gives \(p^{(m,j)} + 1\) fixed points.

Thus we have shown that for \(i = 1\), the number of fixed
points of \(g^j_{c,F}\) is \(\Phi^i_{j}\). The cases \(i = 2, 3, 4, 5\) follow
similarly.

We can use this theorem to calculate the exact formulas
for the number of nontrivial orbits of \(G\) as follows:
For any prime \( p > 2 \) let \( \beta_i = \begin{cases} 1 & \text{if } (-i/p) = 1 \\ 0 & \text{if } (-i/p) = -1 \end{cases} \)

for \( i = 1, 2, 3 \), and \( i \neq p \), and let \( \beta_3 = \begin{cases} 0 & \text{if } 2 \mid m \\ \frac{1}{2} & \text{if } 2 \nmid m \text{ if } p = 3. \end{cases} \)

Then using elementary results of number theory (see [9] Chapter 5) we can summarize the values of \( \beta_i \) as follows:

\[
\begin{array}{ccccccccccc}
p \pmod{24} & 1 & 5 & 7 & 11 & 13 & 17 & 19 & 23 \\
(\beta_1 \beta_2 \beta_3) & 111 & 100 & 001 & 010 & 101 & 110 & 011 & 000 \\
\end{array}
\]

Using this notation we summarize the first few formulas for the number of orbits of \( G \) given \( n = 2m \) in Table I.

**Table I.** Formulas for \( r^*(p,n,n/2) \)

\[
\begin{array}{ccc}
n, n/2 & r^*(p,n,n/2) \\
\hline
2 & 1 \left(\frac{p+1+18\beta_1+12\beta_2+16\beta_3}{24}\right)

4 & 2 \left(\frac{p^2+10p+57+12\beta_1+16\beta_3}{48}\right)

6 & 3 \left(\frac{p^3+18p+19+54\beta_1+36\beta_2+16\beta_3}{72}\right)

8 & 4 \left(\frac{p^4+10p^2+32p+117+32\beta_3}{96}\right)

2q & q \left(\frac{p^q+(q-1)p+q+18q\beta_1+12q\beta_2+16q\beta_3}{24q}\right) \quad (q \text{ prime}, \ q > 3)

12 & 6 \left(\frac{p^6+10p^3+18p^2+36p+171+36\beta_1+16\beta_3}{144}\right)

16 & 8 \left(\frac{p^8+10p^4+32p^2+64p+213+64\beta_3}{192}\right)

18 & 9 \left(\frac{p^9+18p^3+54p+73+162\beta_1+108\beta_2+16\beta_3}{216}\right)

20 & 10 \left(\frac{p^{10}+10p^5+4p^2+40p+285+60\beta_1+80\beta_3}{240}\right)
\end{array}
\]

Using this Table, we can calculate \( r^*(p,n,n/2) \) for small \( p \), and all values of \( r^*(p,n,n/2) \) less than 1000 when \( n > 2 \) are summarized in Table II.
Table II. Values of $r^*(p,n,n/2)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>10</td>
<td>14</td>
<td>42</td>
<td>95</td>
<td>258</td>
<td>672</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>12</td>
<td>27</td>
<td>123</td>
<td>466</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>34</td>
<td>141</td>
<td>850</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>6</td>
<td>22</td>
<td>170</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>8</td>
<td>35</td>
<td>321</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>11</td>
<td>74</td>
<td>907</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>13</td>
<td>101</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>1</td>
<td>17</td>
<td>175</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>2</td>
<td>25</td>
<td>347</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>2</td>
<td>28</td>
<td>422</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>3</td>
<td>38</td>
<td>714</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>3</td>
<td>45</td>
<td>969</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>3</td>
<td>49</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>47</td>
<td>2</td>
<td>57</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>53</td>
<td>3</td>
<td>71</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>59</td>
<td>3</td>
<td>86</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>61</td>
<td>4</td>
<td>92</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>67</td>
<td>4</td>
<td>109</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>71</td>
<td>3</td>
<td>121</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>73</td>
<td>5</td>
<td>128</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>79</td>
<td>4</td>
<td>148</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>83</td>
<td>4</td>
<td>162</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>89</td>
<td>5</td>
<td>185</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>97</td>
<td>6</td>
<td>218</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>101</td>
<td>5</td>
<td>235</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>103</td>
<td>5</td>
<td>244</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>107</td>
<td>5</td>
<td>262</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>109</td>
<td>6</td>
<td>272</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>113</td>
<td>6</td>
<td>291</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>127</td>
<td>6</td>
<td>364</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>131</td>
<td>6</td>
<td>386</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>137</td>
<td>7</td>
<td>421</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>139</td>
<td>7</td>
<td>433</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>149</td>
<td>7</td>
<td>495</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>151</td>
<td>7</td>
<td>508</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>163</td>
<td>8</td>
<td>589</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>167</td>
<td>7</td>
<td>617</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>173</td>
<td>8</td>
<td>661</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>179</td>
<td>8</td>
<td>706</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>181</td>
<td>9</td>
<td>722</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>191</td>
<td>8</td>
<td>801</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>193</td>
<td>10</td>
<td>818</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>197</td>
<td>9</td>
<td>851</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>199</td>
<td>9</td>
<td>868</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>211</td>
<td>10</td>
<td>973</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5. A General Bound

Let \( r^*(p^n) = \min \{ r^*(p,n,m) \mid m \mid n, m \neq n \} \), so \( r^*(p^n) \)
is the least possible number of nontrivial orbits of a maximal solvable primitive non-semilinear subgroup of \( \text{GL}(p,n) \).

Theorem 2.3 \( r^*(p^n) > p^{n/2}/12n \) with the possible exceptions of \( p^n = 17^4, 19^4, 7^6, 5^8, 7^8, 11^8, 13^8, 7^9, 3^{16} \) or \( 5^{16} \). Further when \( n \) is even and \( p \neq 2 \), \( r^*(p^n) \geq r^*(p,n,n/2) \) with the possible exceptions of \( p^n = 3^4, 5^4, 7^4, 11^4, 13^4, 17^4, 19^4, 23^4, 29^4, 31^4, 5^6, 7^6, 3^8, 5^8, 7^8, 11^8, 13^8, 7^9, 3^{12}, 3^{16} \), and \( 5^{16} \), and thus the groups constructed in §2.3 act as a limiting case.

Proof: We wish to show that \( r^*(p,n,m) > p^{n/2}/12n \) for \( m \mid n, m \neq n \), other than for the exceptions above. By (3) it is enough to show that

\[
\frac{(p^n-1) / m(n/m)^2 s(n/m)}{(p^m-1) > p^{n/2}/12n}
\]
or that

\[
p^{n/2-m} > (n/m) s(n/m)/12
\]

(7)

For \( n/m = 2 \), \( s(n/m) = 6 \) and this result is trivially true, so we may assume \( 1 \leq m \leq n/3 \). Suppose first that \( m \geq 5 \) and \( n/m \geq 8 \) (so \( m \leq n/8 \) and \( n \geq 40 \). Then \( m \leq n/8 \) implies that \( p^{n/2-m} > p^{3n/8} \), and \( m \geq 5 \) implies (using (4)) that
\((n/m)s(n/m)/12 < (n/5)^{1+2a}/(24^{1/3}12)\). Thus we want
\(p^{3n/8} > n^{1+2a}/(24^{1/3}5^{1+2a}12)\). This is true for \(n > 40\) and
\(p > 2\). Thus we need now only consider \(m < 5\) or \(n/m < 8\).

Suppose now \(n/m < 8\). From Suprunenko Chapter 21 the
maximal order of a solvable irreducible subgroup of
\(\Omega\)
\(\text{Sp}(2,q) = \text{SL}(2,q)\) is \(\max(2(q+1),s_q)\), where \(s_q\) is 24 if
\(q \equiv \pm 1 (\mod 8)\) and 48 otherwise, and the maximal order of
a reducible subgroup of \(\text{SL}(2,q)\) is \(q(q-1)\). Further by
Dixon [2] the maximal order of a solvable subgroup of
\(\text{Sp}(2,4)\) is 72. Thus we have the following upper bounds
for \(s(n/m)\):

\[
\begin{array}{ccccccc}
  n/m & 2 & 3 & 4 & 5 & 6 & 7 \\
  s(n/m) & 6 & 24 & 72 & 24 & 144 & 48 \\
\end{array}
\]

So, considering both sides of (7), we can summarize the
possible \(p^n\) for which (7) may not hold as follows:

\[
\begin{array}{cccc}
  m & p^{n/2-m} & (n/m)s(n/m)/12 & \text{Possible Exceptions} \\
  \hline
  n/3 & p^{n/6} & 6 & 2^3, 3^3, 5^3, \ldots, 31^3 \\
  & & & \ldots, 2^{12}

  n/4 & p^{n/4} & 24 & 2^4, 3^4, 5^4, \ldots, 23^4 \\
  & & & \ldots, 2^{16}

  n/5 & p^{3n/10} & 10 & 2^5, 3^5, 2^{10}

  n/6 & p^{n/3} & 72 & 2^6, 3^6, 5^6, 7^6, 2^{12}, 2^{18}

  n/7 & p^{5n/14} & 28 & 2^7, 3^7
\end{array}
\]
But by Suprunenko Corollary 20.3.1 every prime divisor of 
\( \frac{n}{m} \) must divide \( \frac{p^m-1}{p^m-1} \). This eliminates all of the above exceptions except 
\( 7^3, 13^3, 19^3, 31^3, 2^6, 5^6, 2^{12} \) (m = n/3), 
\( 3^4, 5^4, 7^4, 11^4, 13^4, 17^4, 19^4, 23^4, 3^8 \) (m = n/4) and \( 7^6 \) (m = n/6).

Further by Foulser [4] Theorem 6.34,

\[
r^*(p,3,1) = \frac{(p^2 + 34p + 361 + 144\delta_1 + 108\delta_2)}{216}
\]  

where \( \delta_1 = \begin{cases} 1 & \text{if } p \equiv 1 (\mod 9) \\ 0 & \text{otherwise} \end{cases} \) and \( \delta_2 = \begin{cases} 1 & \text{if } (3/p) = 1 \\ 0 & \text{otherwise} \end{cases} \)

And Theorem 1.1 and Theorem 1.2 of Foulser give the following lower bounds for \( r^*(p^n) \):

<table>
<thead>
<tr>
<th>( p^n )</th>
<th>2^6</th>
<th>5^6</th>
<th>2^{12}</th>
<th>3^4</th>
<th>5^4</th>
<th>7^4</th>
<th>11^4</th>
<th>13^4</th>
<th>3^8</th>
<th>7^6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r^*(p^n) )</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Thus we can eliminate all of the possible exceptions except 
\( 7^6, 17^4, 19^4 \) and \( 23^4 \). Finally we can eliminate \( 23^4 \) using (3), leaving the three possible exceptions \( 7^6, 17^4 \) and \( 19^4 \).

So now we suppose \( m = 1, 2, 3 \) or \( 4 \) and \( n \geq 8m \). Using (4) we want

\[
p^{n/2-m} > n^{1+2\alpha} / (24^{1/3} \cdot 12^m 1+2\alpha)
\]  

For fixed \( m \) let \( f(n,p) = p^{n/2-m} - n^{1+2\alpha} / (24^{1/3} \cdot 12^m 1+2\alpha) \), so (9) is true if and only if \( f(n,p) > 0 \). Then for fixed \( n \) \( f(n,p) \) increases with \( p \), and for fixed \( p \) \( f(n,p) \) increases with \( n \) provided either \( p > 5 \), or \( p > 3 \) and \( n > 10 \), or \( p > 2 \) and \( n > 16 \). Thus we can summarize the possible
exceptions as follows:

<table>
<thead>
<tr>
<th>m</th>
<th>( p^n ) such that ( f(n,p) &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 2^8, 2^9, \ldots, 2^{55}, 3^8, 3^9, \ldots, 3^{29}, 5^8, 5^9, \ldots, 5^{16} )</td>
</tr>
<tr>
<td></td>
<td>( 7^8, 7^9, \ldots, 7^{12}, 11^8, 11^9, 13^8 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2^{16}, 2^{18}, \ldots, 2^{40}, 3^{16}, 3^{18}, \ldots, 3^{20} )</td>
</tr>
<tr>
<td>3</td>
<td>( 2^{24}, 2^{27}, 2^{30}, 2^{33} )</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
</tr>
</tbody>
</table>

If we test each of these cases as above, the only new exceptions that we find are \( 5^8, 7^8, 11^8, 13^8, 7^9, 3^{16}, 5^{16} \). Thus we have proved \( r^*(p^n) > p^{n/2}/12n \) with ten possible exceptions.

Finally, note that by Theorem 2.2,

\[
p^{n/2}/12n < r^*(p,n,n/2) < p^{n/2}/6n
\]

for all \( n > 9, p > 2 \). Thus \( r^*(p^n) \geq r^*(p,n,n/2) \) can be proved as above.

Now by using this theorem, Table I, formula (8), and bounds (3) and (4) we can calculate all possible values of \( p \) and \( n \) such that there may exist a maximal solvable primitive subgroup of \( \text{GL}(p,n) \) with at most 1000 nontrivial orbits, and these results are summarized in Table III.
Table III. Values of \( r^*(p^n) \) less than 1000

(i) \( n = 2 \).

\[
\begin{align*}
& \leq 1000 \text{ for all odd primes } p \leq 23957 \\
& r^*(p^2) = 1000 \text{ for } p = 23967, 23981 \\
& > 1000 \text{ for } p = 23977 \text{ and } p \geq 23993.
\end{align*}
\]

The exact formula for \( r^*(p^2) = r^*(p, 2, 1) \) is in Table I.

(ii) \( n > 2 \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( n )</th>
<th>( r^* )</th>
<th>( p )</th>
<th>( n )</th>
<th>( r^* )</th>
<th>( p )</th>
<th>( n )</th>
<th>( r^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>10</td>
<td>27</td>
<td>79</td>
<td>4</td>
<td>148</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>2</td>
<td>61</td>
<td>3</td>
<td>33</td>
<td>163</td>
<td>3</td>
<td>151</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>2</td>
<td>2</td>
<td>20</td>
<td>30</td>
<td>83</td>
<td>4</td>
<td>162</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>2</td>
<td>67</td>
<td>3</td>
<td>33</td>
<td>23</td>
<td>6</td>
<td>175</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>3</td>
<td>13</td>
<td>6</td>
<td>35</td>
<td>181</td>
<td>3</td>
<td>183</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>3</td>
<td>37</td>
<td>4</td>
<td>38</td>
<td>89</td>
<td>4</td>
<td>185</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>73</td>
<td>3</td>
<td>38</td>
<td>193</td>
<td>3</td>
<td>205</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>24</td>
<td>39</td>
<td>199</td>
<td>3</td>
<td>217</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>4</td>
<td>79</td>
<td>3</td>
<td>43</td>
<td>97</td>
<td>4</td>
<td>218</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>21</td>
<td>43</td>
<td>101</td>
<td>4</td>
<td>235</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>4</td>
<td>41</td>
<td>4</td>
<td>45</td>
<td>211</td>
<td>3</td>
<td>241</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>4</td>
<td>43</td>
<td>4</td>
<td>49</td>
<td>103</td>
<td>4</td>
<td>244</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>57</td>
<td>20</td>
<td>3</td>
<td>258</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>4</td>
<td>97</td>
<td>3</td>
<td>61</td>
<td>107</td>
<td>4</td>
<td>262</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>4</td>
<td>103</td>
<td>3</td>
<td>67</td>
<td>223</td>
<td>3</td>
<td>267</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>4</td>
<td>53</td>
<td>4</td>
<td>71</td>
<td>109</td>
<td>4</td>
<td>272</td>
</tr>
<tr>
<td>11</td>
<td>4</td>
<td>4</td>
<td>17</td>
<td>6</td>
<td>74</td>
<td>13</td>
<td>8</td>
<td>272</td>
</tr>
<tr>
<td>13</td>
<td>4</td>
<td>4</td>
<td>109</td>
<td>3</td>
<td>75</td>
<td>229</td>
<td>3</td>
<td>281</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
<td>5</td>
<td>59</td>
<td>4</td>
<td>86</td>
<td>113</td>
<td>4</td>
<td>291</td>
</tr>
<tr>
<td>17</td>
<td>4</td>
<td>5</td>
<td>11</td>
<td>8</td>
<td>86</td>
<td>241</td>
<td>3</td>
<td>309</td>
</tr>
<tr>
<td>19</td>
<td>4</td>
<td>6</td>
<td>61</td>
<td>4</td>
<td>92</td>
<td>29</td>
<td>6</td>
<td>347</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>6</td>
<td>3</td>
<td>18</td>
<td>95</td>
<td>127</td>
<td>4</td>
<td>364</td>
</tr>
<tr>
<td>19</td>
<td>3</td>
<td>7</td>
<td>127</td>
<td>3</td>
<td>97</td>
<td>271</td>
<td>3</td>
<td>385</td>
</tr>
<tr>
<td>31</td>
<td>3</td>
<td>11</td>
<td>19</td>
<td>6</td>
<td>101</td>
<td>131</td>
<td>4</td>
<td>386</td>
</tr>
<tr>
<td>23</td>
<td>4</td>
<td>11</td>
<td>67</td>
<td>4</td>
<td>109</td>
<td>277</td>
<td>3</td>
<td>401</td>
</tr>
<tr>
<td>7</td>
<td>9</td>
<td>13</td>
<td>139</td>
<td>3</td>
<td>113</td>
<td>283</td>
<td>3</td>
<td>417</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>14</td>
<td>71</td>
<td>4</td>
<td>121</td>
<td>137</td>
<td>4</td>
<td>421</td>
</tr>
<tr>
<td>37</td>
<td>3</td>
<td>15</td>
<td>5</td>
<td>12</td>
<td>123</td>
<td>31</td>
<td>6</td>
<td>422</td>
</tr>
<tr>
<td>43</td>
<td>4</td>
<td>17</td>
<td>73</td>
<td>4</td>
<td>128</td>
<td>139</td>
<td>4</td>
<td>433</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>22</td>
<td>151</td>
<td>3</td>
<td>131</td>
<td>5</td>
<td>14</td>
<td>466</td>
</tr>
<tr>
<td>29</td>
<td>4</td>
<td>22</td>
<td>7</td>
<td>10</td>
<td>141</td>
<td>2</td>
<td>30</td>
<td>486</td>
</tr>
<tr>
<td>31</td>
<td>4</td>
<td>26</td>
<td>157</td>
<td>3</td>
<td>141</td>
<td>307</td>
<td>3</td>
<td>487</td>
</tr>
</tbody>
</table>

†Lower Estimate
Chapter III. The Imprimitive Case

1. The Wreath Product

The material in this section is primarily from Suprunenko [14] Chapters 3 and 15.

Let $U$ be a linear space over $GF(p)$, $H$ a subgroup of $GL(U)$, and $K$ a subgroup of the symmetric group $S_k$, $k > 1$.
Let $V$ be the cartesian product $U^k$ as a linear space over $GF(p)$, so we can write $V = \{(u_1, \ldots, u_k) \mid u_i \in U\}$.
Let $G = \{(g_1, \ldots, g_k; s) \mid g_i \in H$ and $s \in K\}$, and for $g = (g_1, \ldots, g_k; s)$ in $G$, define $g: V \to V$ by

$$g((u_1, \ldots, u_k)) = (u_1^g, \ldots, u_k^g), \text{ for } u_i^g = g_{s^{-1}(i)}(u_{s^{-1}(i)})$$

For $f = (f_1, \ldots, f_k; r)$ in $G$ we have

$$gf = (g_{r(1)}f_1, \ldots, g_{r(k)}f_k; sr) \in G$$

and so $G$ is a subgroup of $GL(V)$. $G$ is called the wreath product of the linear group $H$ and the permutation group $K$ and is denoted $G = H \wr K$.

Theorem 3.1 Let $H$ be an irreducible subgroup of $GL(U)$ and $K$ a transitive subgroup of $S_k$, $k > 1$. Then the group $G = H \wr K$ is imprimitive, and is irreducible except when $U$ is a one-dimensional space over $GF(2)$.

Proof: Suprunenko Lemma 15.4.
Now suppose $G$ is an irreducible imprimitive subgroup of $GL(V)$, for $V$ a linear space over $GF(p)$, and let $V = U_1 + \ldots + U_k$ be a system of imprimitivity for $G$. Then we can define a mapping $\psi: G \rightarrow S_k$ by $\psi(g): i \mapsto j$, where $g(U_i) = U_j$, for any $g$ in $G$. Let $K = \psi(G)$. Then $K$ is a transitive subgroup of $S_k$. Now let $G_1 = \{ g \in G \mid g(U_1) = U_1 \}$, and let $H = G_1|U_1$. Then we have the following theorem:

**Theorem 3.2** $G$ is similar as a linear group to a subgroup of the wreath product $H \wr K$.

**Proof:** Suprunenko Lemma 15.5.

This construction leads to the following theorem:

**Theorem 3.3** Let $G$ be an imprimitive irreducible maximal solvable subgroup of $GL(V)$, where $V$ is a linear space of finite dimension over $GF(p)$. Then $G$ is similar as a linear group to a wreath product $H \wr K$, where $H$ is a maximal primitive solvable subgroup of $GL(U)$, $K$ is a maximal transitive solvable subgroup of the symmetric group $S_K$, and $U$ is a subspace of $V$ such that $[U:GF(p)]k = n$, for $n = [V:GF(p)]$.

**Proof:** Suprunenko Theorem 15.5.

Paralleling the construction of the wreath product of a linear group and a permutation group is the construction
of the wreath product of two permutation groups. Let \( M, N \) be subgroups of \( \text{Sym}(X) \), \( \text{Sym}(Y) \) respectively, where \(|X| = m\) and \(|Y| = n\). Let \( Z = X \times Y \) and let \( K = \{(r_1, \ldots, r_m; s) \mid r_j \in M, s \in N\} \). For \( r = (r_1, \ldots, r_m; s) \) define \( r: Z \to Z \) by
\[
x, y) = (r_y(x), s(y))
\]
For \( t = (t_1, \ldots, t_m; s') \) in \( K \) we have
\[
rt = (r_{s'}(l)t_1, \ldots, r_{s'}(m)t_m; ss') \text{ in } K
\]
and so \( K \) is a subgroup of \( \text{Sym}(Z) \). \( K \) is called the wreath product of \( M \) and \( N \) in its imprimitive action and is again denoted \( K = M \text{ wr } N \). And as above we have the following theorem:

**Theorem 3.4** Let \( K \) be a maximal solvable imprimitive subgroup of \( S_k \). Then \( K \) is similar as a permutation group to a wreath product \( M \text{ wr } N \), where \( M \) is a maximal solvable primitive subgroup of \( S_m \), \( N \) is a maximal solvable transitive subgroup of \( S_n \), and \( k = mn \).

These wreath products can be combined, and in fact the combinations are all associative.

**Theorem 3.5** Let \( K_i \) be a subgroup of \( \text{Sym}(X_i) \), \( i = 1, 2, 3 \). Then \((K_1 \text{ wr } K_2) \text{ wr } K_3 \) and \( K_1 \text{ wr } (K_2 \text{ wr } K_3) \) are similar.

**Proof:** Huppert [6] Lemma 15.4.
Theorem 3.6 Let \( H \) be a subgroup of \( GL(U) \) and let
\( M, N \) be subgroups of \( Sym(X) \) and \( Sym(Y) \) respectively, where
\( |X| = m \) and \( |Y| = n \). Then \( H \ wr (M \ wr N) \) and \( (H \ wr M) \ wr N \)
are similar.

Proof: Let \( G_1 = (H \ wr M) \ wr N \) and \( G_2 = H \ wr (M \ wr N) \).
Then \( G_1 \) acts on \( (U^m)^n \) and \( G_2 \) acts on \( U^{(mn)} \) so we identify
\( (U^m)^n \) and \( U^{(mn)} \) via \( v = ((v_{11}, \ldots, v_{1m}), \ldots, (v_{n1}, \ldots, v_{nm})) \)
\( = (v_{11}, \ldots, v_{1m}, \ldots, v_{n1}, \ldots, v_{nm}) \). Let \( g \in G_1 \), say
\( g = (f_1, \ldots, f_n;t) \) for \( t \) in \( N \) and \( f_i = (h_{i1}, \ldots, h_{im}, s_i) \) in
\( H \ wr M, h_{ij} \) in \( H \) and \( s_i \) in \( M \). Define \( \psi: G_1 \to G_2 \) by
\[
\psi(g) = (h_{i1}, \ldots, h_{im}, \ldots, h_{n1}, \ldots, h_{nm}; r)
\]
for \( r = (s_1, \ldots, s_n; t) \) in \( M \ wr N \). Then for any \( v \) in \( (U^m)^n \),
\( v = (w_1, \ldots, w_n) \) with \( w_i = (v_{i1}, \ldots, v_{im}) \), we have
\[
g(v) = (w'_1, \ldots, w'_n) \quad \text{for} \quad w'_i = f_{t^{-1}(i)}'(w_{t^{-1}(i)}).
\]
But then \( w'_i = (v'_{i1}, \ldots, v'_{im}) \) for \( v'_{ij} = h_{ij}'(v_{ij}) \), where
\( i' = t^{-1}(i) \) and \( j' = (s_{t^{-1}(j)})^{-1}(j) \). Also we have
\[
\psi(g)(v) = (h_{r^{-1}(1)}, (v_{r^{-1}(1)}), \ldots, h_{r^{-1}(n)}, (v_{r^{-1}(n)})).
\]
But \( r(i'j') = (s_1, \ldots, s_n; t)(i'j') \)
\[
= t(i')s_{i'}(j') = t(t^{-1}(i))s_{i'}(s_{t^{-1}(j)})^{-1}(j)
\]
\( = ij \).
Thus $r^{-1}(ij) = i'j'$, and so $h^{-1}(v^{-1}(ij)) = h_{i'j'}(v_{i'j'})$

$= v_{ij}$. Therefore $\psi(g)(v) = g(v)$, and so $G_1$ and $G_2$ are similar as linear groups.

Thus any maximal solvable imprimitive subgroup $G$ of $GL(n,p)$ can be written $G = H \mathrm{wr} K_1 \mathrm{wr} K_2 \mathrm{wr} \ldots \mathrm{wr} K_s$, where $H$ is a maximal solvable primitive subgroup of $GL(n_1,p)$, $K_i$ is a maximal solvable primitive subgroup of $S_{k_i}$, $i=1,\ldots,s$, and $n = n_1k_1k_2\ldots k_s$. Further, by associativity, $G = (H \mathrm{wr} K_1 \mathrm{wr} K_2 \mathrm{wr} \ldots \mathrm{wr} K_{s-1}) \mathrm{wr} K_s$, and so any maximal solvable imprimitive subgroup $G$ of $GL(n,p)$ can be written in the form $G = H' \mathrm{wr} K$, where $H'$ is a maximal solvable irreducible subgroup of $GL(n_2,p)$, $K$ is a maximal solvable primitive subgroup of $S_k$, and $n = n_2k$.

2. The Number of Orbits of a Wreath Product

Theorem 3.7 Let $G$ be a maximal solvable imprimitive subgroup of $GL(n,p)$, with $G = H \mathrm{wr} K$, where $H$ is an irreducible solvable subgroup of $GL(m,p)$ with $h$ orbits and $K$ is a maximal solvable primitive subgroup of $S_k$, $m = km$. Let $r'$ be the number of orbits of $G$. Then

$$r' = \frac{1}{|K|} \sum_{x \in K} h^c(x)$$

where $c(x)$ is the number of cycles in $x$ for each permutation $x$ in $K$. 
Proof: Let $P = GF(p)$ and let $V = P^{(n)}$, so that $GL(n, p) = GL(V)$. Let $V = V_1 + \ldots + V_k$ be the imprimitivity decomposition corresponding to the wreath product $G = H \wr K$ so each $V_i$ is isomorphic to $U = P^{(m)}$, and $GL(m, p) = GL(U)$. Let $R_1, \ldots, R_h$ be the orbits of $H$ on $U$, and let $R_{i_1}, \ldots, R_{i_h}$ be the corresponding orbits on $V_i$, $i = 1, \ldots, k$. Let $R(j_1, \ldots, j_k) = R_{j_1} + \ldots + R_{j_k}$ for each $(j_1, \ldots, j_k)$ in $X^k$, $X = \{1, \ldots, h\}$. Then each $R(j_1, \ldots, j_k)$ is wholly contained in an orbit of $H \wr K$, any two such are disjoint, and the union of all such sets is $V$. Define an action of $K$ on $X^k$ by $s:(j_1, \ldots, j_k) + (i_1, \ldots, i_k)$ if and only if $(1, \ldots, 1; s)(R(j_1, \ldots, j_k)) = R(i_1, \ldots, i_k)$, where $(1, \ldots, 1; s) \in H \wr K$. Then $K$ is equivalent to a subgroup of $Sym(X^k)$ and $(j_1, \ldots, j_k), (i_1, \ldots, i_k)$ are in the same orbit of $K$ on $X^k$ if and only if $R(j_1, \ldots, j_k), R(i_1, \ldots, i_k)$ are in the same orbit of $H \wr K$. Thus the number of orbits of $G$ on $V$ is equal to the number of orbits of $K$ on $X^k$, and so by Burnside's Lemma we have

$$r' = \frac{1}{|K|} \sum_{x \in K} |Fix(x)|$$

But $Fix(x) = \{(j_1, \ldots, j_k) \in X^k \mid \text{all } j_i \text{ with } i \text{ in the same cycle of } x \text{ are equal}\}$, for any $x$ in $K$. Thus we have

$$|Fix(x)| = |X|^c(x) = h^c(x)$$

and thus substituting this in the above equation proves the theorem.
As a result of this theorem note that for any permutation group $K$, if $H$ and $H'$ have the same number of orbits then $H \wr K$ and $H' \wr K$ will have the same number of orbits. Thus we can define $(h)K$ to be the number of orbits of $H \wr K$ for any linear group $H$ having $h$ orbits. We shall now compute $(h)K$ for certain $K$.

3. Some Small Cases

Following the notation of Pálfy [10], let $R_q$ denote the group of linear transformations $x \rightarrow ax + b$, for $a, b$ in $GF(q)$, $a \neq 0$, and $q$ prime, and let $R_4 = S_4$. Then by Suprunenko [14] Theorem 4.11, $R_q$ is the unique maximal solvable transitive subgroup of $S_q$ and $R_q$ is primitive. Further $|R_q| = q(q-1)$ for $q$ prime, $|R_4| = 24$, and $R_2 = S_2$, $R_3 = S_3$. Then considering the cycle structure of $R_q$ for $q \geq 5$ we find one element with $q$ cycles, $(q-1)$ elements with one cycle, and for each $k | (q-1)$, $\phi(k)q$ elements with $1 + (q-1)/k$ cycles (where $\phi(k)$ here is the Euler function). Thus by Theorem 3.7 we have

$$
(h)R_q = (h^q + (q-1)h + \sum_{k | (q-1)} q\phi(k)h^{(1+(q-1)/k)})/(q(q-1))
$$

(1)

Further by Foulser [4] Lemma 2.6 we have

$$
(h)R_k = \binom{k+h-1}{k} \text{ for } k = 2, 3, 4
$$

(2)
Now suppose $K$ is a maximal solvable primitive subgroup of $S_8$. Since the stabilizer of $K$ must be an irreducible subgroup of $GL(3,2)$, the only possibility ([14] Chapter 21) is the semilinear group $S(2^3)$. So let $R_8 = S(2^3)$. Then $R_8$ can be generated by the elements $(2\ 3\ 4\ 5\ 6\ 7\ 8)$, $(3\ 4\ 6)(5\ 8\ 7)$ and $(1\ 2)(3\ 5)(4\ 8)(6\ 7)$, and so the cycle structure of the elements of $R_8$ is as follows:

<table>
<thead>
<tr>
<th>Structure</th>
<th>Number of Elements</th>
<th>Number of Cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-7</td>
<td>48</td>
<td>2</td>
</tr>
<tr>
<td>2-6</td>
<td>56</td>
<td>2</td>
</tr>
<tr>
<td>1-1-3-3</td>
<td>56</td>
<td>4</td>
</tr>
<tr>
<td>2-2-2-2</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>identity</td>
<td>1</td>
<td>8</td>
</tr>
</tbody>
</table>

Thus $(h)R_8 = (h^8 + 63h^4 + 104h^2)/168$.

Now suppose $K$ is a maximal solvable primitive subgroup of $S_9$. Here ([14] Chapter 21) there are three possibilities for the stabilizer of $K$ as a maximal solvable irreducible subgroup of $GL(2,3)$, and so we consider each of these individually:

(i) $K_1$, the semilinear group $S(3^2)$. Then $K_1$ has 144 elements and is generated by the elements $(4\ 7)(5\ 8)(6\ 9)$, $(2\ 5\ 7\ 8\ 3\ 9\ 4\ 6)$, and $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$, and so the cycle structure of the elements of $K_1$ is as follows:
Thus \((h)K_1 = \left(h^9 + 12h^6 + 9h^5 + 62h^3 + 60h^2\right)/144\).

(ii) \(K_2\) with imprimitive stabilizer. Then \(K_2\) has 72 elements and is generated by the elements \((2\ 3)(5\ 6)(8\ 9)\), \((2\ 4)(3\ 7)(6\ 8)\) and \((1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)\), and so the cycle structure of the elements of \(K_2\) is as follows:

Thus \((h)K_2 = \left(h^9 + 12h^6 + 9h^5 + 26h^3 + 24h^2\right)/72.\)

<table>
<thead>
<tr>
<th>Structure</th>
<th>Number of Elements</th>
<th>Number of Cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-8</td>
<td>36</td>
<td>2</td>
</tr>
<tr>
<td>6-3</td>
<td>24</td>
<td>_2</td>
</tr>
<tr>
<td>1-4-4</td>
<td>54</td>
<td>3</td>
</tr>
<tr>
<td>3-3-3</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>1-2-2-2-2</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>1-1-1-2-2-2</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>identity</td>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Structure</th>
<th>Number of Elements</th>
<th>Number of Cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-3</td>
<td>24</td>
<td>2</td>
</tr>
<tr>
<td>4-4-1</td>
<td>18</td>
<td>3</td>
</tr>
<tr>
<td>3-3-3</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>2-2-2-2-1</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>1-1-1-2-2-2</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>identity</td>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>
(iii) $K_3$ with stabilizer having the Suprunenko series $v^6 \cdot A^4 \cdot F^2 \cdot 1$. Then $K_3$ has 432 elements and is generated by the elements $(2 \ 3)(4 \ 6)(7 \ 8)$, $(2 \ 8 \ 4 \ 5 \ 3 \ 6 \ 7 \ 9)$ and $(1 \ 2 \ 3)(4 \ 5 \ 6)(7 \ 8 \ 9)$, and so the cycle structure of the elements of $K_3$ is as follows:

<table>
<thead>
<tr>
<th>Structure</th>
<th>Number of Elements</th>
<th>Number of Cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-1</td>
<td>108</td>
<td>2</td>
</tr>
<tr>
<td>6-3</td>
<td>72</td>
<td>2</td>
</tr>
<tr>
<td>6-3-1</td>
<td>72</td>
<td>3</td>
</tr>
<tr>
<td>4-4-1</td>
<td>54</td>
<td>3</td>
</tr>
<tr>
<td>3-3-3</td>
<td>56</td>
<td>3</td>
</tr>
<tr>
<td>3-3-1-1-1</td>
<td>24</td>
<td>5</td>
</tr>
<tr>
<td>2-2-2-2-1</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>2-2-2-1-1-1</td>
<td>36</td>
<td>6</td>
</tr>
<tr>
<td>identity</td>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>

Thus $(h)K_3 = (h^9 + 36h^6 + 33h^5 + 182h^3 + 180h^2)/432$.

Since $(h)K_3$ is less than either $(h)K_1$ or $(h)K_2$ for any $h$, we let $R_9 = K_3$. Then for $k = 4, 8, 9, \text{ or } q$, for any prime $q$, we have an exact formula for the number of orbits of $H \wr R_k$ whenever $H$ is an irreducible subgroup of $GL(m,p)$ with $h$ orbits. These formulas are summarized for values of $k$ up to 23 in Table IV.
Table IV. Formulas for \( (h)R_k \)

<table>
<thead>
<tr>
<th>( j )</th>
<th>( (h)R_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( (h^2 + h)/2 )</td>
</tr>
<tr>
<td>3</td>
<td>( (h^3 + 3h^2 + 2h)/6 )</td>
</tr>
<tr>
<td>4</td>
<td>( (h^4 + 6h^3 + 11h^2 + 6h)/24 )</td>
</tr>
<tr>
<td>5</td>
<td>( (h^5 + 5h^3 + 10h^2 + 2h)/20 )</td>
</tr>
<tr>
<td>7</td>
<td>( (h^7 + 7h^4 + 14h^3 + 14h^2 + 6h)/42 )</td>
</tr>
<tr>
<td>8</td>
<td>( (h^8 + 6h^4 + 10h^2)/168 )</td>
</tr>
<tr>
<td>9</td>
<td>( (h^9 + 36h^6 + 33h^5 + 182h^3 + 180h^2)/432 )</td>
</tr>
<tr>
<td>11</td>
<td>( (h^{11} + 11h^6 + 44h^3 + 44h^2 + 10h)/110 )</td>
</tr>
<tr>
<td>13</td>
<td>( (h^{13} + 13h^7 + 26h^5 + 26h^4 + 26h^3 + 52h^2 + 12h)/156 )</td>
</tr>
<tr>
<td>17</td>
<td>( (h^{17} + 17h^9 + 34h^5 + 68h^3 + 136h^2 + 16h)/272 )</td>
</tr>
<tr>
<td>19</td>
<td>( (h^{19} + 19h^{10} + 38h^7 + 38h^4 + 114h^3 + 114h^2 + 18h)/342 )</td>
</tr>
<tr>
<td>23</td>
<td>( (h^{23} + 23h^{12} + 230h^3 + 230h^2 + 22h)/506 )</td>
</tr>
</tbody>
</table>

Using the above formulas, we can calculate all values of \( (h)R_k \) which are less than or equal to 1000, and these are summarized in Table V.

Note that if \( H \) is an irreducible linear group with \( h \) orbits, then \( H \wr R_{k_1} \) is an irreducible linear group with \( (h)R_{k_1} \) orbits, and thus since wreath products are associative, \( (h)(R_{k_1} \wr R_{k_2}) = ((h)R_{k_1})R_{k_2} \). Thus we can calculate the number of orbits for strings of wreath products; e.g. \( (2)(R_8 \wr R_3) = ((2)R_8)R_3 = (10)R_3 = 220 \).
Table V. Values of $(h)R_t$ at most 1000

<table>
<thead>
<tr>
<th>$t$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>14</td>
<td>30</td>
<td>74</td>
<td>522</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>24</td>
<td>78</td>
<td>75</td>
<td>140</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>20</td>
<td>35</td>
<td>76</td>
<td>460</td>
<td>496</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>35</td>
<td>70</td>
<td>201</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>21</td>
<td>56</td>
<td>126</td>
<td>462</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>28</td>
<td>84</td>
<td>210</td>
<td>952</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>36</td>
<td>120</td>
<td>330</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>45</td>
<td>165</td>
<td>495</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>55</td>
<td>220</td>
<td>715</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>66</td>
<td>286</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>78</td>
<td>364</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>91</td>
<td>455</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>105</td>
<td>560</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>120</td>
<td>680</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>136</td>
<td>816</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>153</td>
<td>969</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>171</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>190</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>210</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>231</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>253</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>276</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>300</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>325</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>351</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>378</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>406</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>435</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>465</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>496</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>528</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>561</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>595</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>630</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>666</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>703</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>741</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>780</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>820</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>861</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>903</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>946</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>44</td>
<td>990</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4. The Case $K$ is Primitive

Theorem 3.8 Let $K$ be a maximal solvable primitive subgroup of $S_k$, where $k = q^s$, $q$ prime, $s > 1$. Then there exist transitive solvable subgroups $K_i$ of $S_{k_i}$, $i = 1, 2$ such that $k = k_1k_2$ and $(h)(K_1 \wr K_2) < (h)K$, except when $k = 4$, or $k = 8$ and $h = 2, 3, 4$, or $k = 9$ and $h = 2, 3, 4, 5, 6$.

Proof: Let $K_i = R_q^{(i)} = R_q \wr R_q \wr \ldots \wr R_q$ for $i = 0, 1, \ldots, s$, and let $h_i = (h)K_i$, so $h_0 = h$ and $h_{i+1} = (h_i)R_q$ for $i = 0, 1, \ldots, s-1$. We will show that $(h)(K_{s-1} \wr R_q) < (h)K$.

By Palfy [10] Theorem 1 we have

$$|K| \leq 24^{1/3} (k^{C_q})$$

where

$$c_2 = 1 + \log_4 6 \cdot 24^{1/3} = 3.05664 \ldots,$$
$$c_3 = 1 + \log_9 48 \cdot 24^{1/3} = 3.24399 \ldots,$$
$$c_5 = 1 + \log_{25} 96 \cdot 24^{1/3} = 2.74710 \ldots,$$
$$c_7 = 1 + \log_{49} 144 \cdot 24^{1/3} = 2.54918 \ldots,$$
$$c_q = 1 + \log_q (q-1) \cdot 24^{1/3} < 2.5 \text{ for } q > 7.$$

Thus by Theorem 3.7 we have

$$(h)K > \frac{1}{|K|} h^k \geq 24^{1/3} h^{k/(k^{C_q})}$$  (3)
Suppose first that \( q \geq 11 \). Then by (1) we have
\[
\frac{h_{i+1}}{h_i} \leq \frac{q}{q(q-1)} \quad i=0,1,\ldots,s-1,
\]
and so
\[
h_s < \frac{2}{q(q-1)} \frac{(q^{s-1})/(q-1)}{h(q^s)}
\]  
(4)
But \( h_s = (h)(K_{s-1} \wr R_q) \), and so by combining (3) and (4) we have \((h)K > (h)(K_{s-1} \wr R_q)\) if
\[
\frac{(q(q-1))(q^{s-1})/(q-1)}{2(q^{s-1})/(q-1)(q^s)} < \frac{c_q}{24^{1/3}}
\]
or if
\[
q^{(s-1)} > q^{2.5s},
\]
which is true for all \( s > 1, q > 7 \).

Now suppose \( q = 7 \). Then by (1), \( h_{i+1} < h_i^{7/12} \), and using this as above gives the same result. Similarly for \( q = 5 \) we can use \( h_{i+1} < h_i^{5/5} \).

Now let \( q = 3 \). Then (1) gives \( h_{i+1} < h_i^{3/2} \), and so
\[
(h)K > (h)(K_{s-1} \wr R_3) \quad \text{if}
\]
\[
\frac{(1/2)(3^{s-1})/2}{24^{1/3}} < \frac{3.244s}{(3.244s)}
\]
which is true for all \( s > 3 \). Further for \( h > 10 \), (1) gives
\[
h_{i+1} < 286h_i^{3}/1331,
\]
and as above this gives
\((h)K > (h)(K_{s-1} \wr R_3)\) for \( s > 1 \). Thus we need only consider \( s = 2,3 \) and \( h = 2,3,\ldots,10 \). But then using (3) and the exact formula for \((h)(R_3 \wr R_3 \wr R_3)\) from (2) we can eliminate the case \( s = 3 \). For \( s = 2 \), the three possibilities for \( K \) were discussed in \( \S 3.3 \); using the exact formulas for these \( K \) and that for \((h)(R_3 \wr R_3)\) eliminates the cases \( s = 2, h = 7,8,9,10 \), and in fact we have
(h) $R_9 < (h)(R_3 \wr R_3)$ for $h = 2, 3, 4, 5, 6$. This completes the case $q = 3$.

Finally, suppose $q = 2$. Then we consider separately five possibilities for $s$:

(i) $s = 2$. Then using the exact formulas from (2) we have $(h)R_4 < (h)(R_2 \wr R_2)$ for all $h > 2$.

(ii) $s = 3$. Then as in §3.3, the only possibility for $K$ is $R_8$. Again using the exact formulas, we have $(h)(R_2 \wr R_4) < (h)R_8$ for $h > 4$, and in fact $(h)R_8 < (h)(R_2 \wr R_4) < (h)(R_4 \wr R_2)$ for $h = 2, 3, 4$.

(iii) $s = 4$. Then by Suprunenko [14] Chapter 21 there are only two possibilities for $K$: the semilinear group $S(2^4)$ of order 960 and the group with imprimitive stabilizer of order 1152. Thus $(h)K > h^{16}/1152$. But $(h)(R_4 \wr R_4) < (5/27)^5 h^{16}$, and so $(h)K > (h)(R_4 \wr R_4)$ for all $h > 2$. For $h = 2$, $(h)(R_8 \wr R_2) = 55$, and $(h)K > 2^{16}/1152 > 56$, and thus the result is true.

(iv) $s$ odd, $s > 3$. Then by (2) $(h)R_2 < h^2$ and $(h)R_4 < 5h^4/16$, and so we have

$$(h)(R_4 ((s-1)/2) \wr R_2) < [(5/16)(4(s-1)/2-1)/3] h^{2s}$$

and $(h)K > 2^{1/3} h^{2s} / (2^{3.06s})$. Thus we want
\[(16/5)^{2} \left( 4^{(s-1)/2}-1 \right)/3 > 2^{3.06s}/24^{1/3}\]

which is true for \(s \geq 5\).

(v) \(s\) even, \(s > 4\). Then as above we have

\[(h)(R_{4}^{(s/2)}) < (5/16)(4^{s/2}-1)/3h^{2s}\]

and so as above we get \((h)K > (h)(R_{4}^{((s/2)-1)_{WR} R_{4}})\).

This completes the proof of the theorem.

Now define \(r(k,h) = \min \{(h)K \mid K \text{ is a transitive solvable subgroup of } S_{k}\}\). Then, by the above theorem and the associativity of wreath products, \(r(k,h) = (h)K_{1}\) for some \(K\) of the form \(K = R_{k_{1}} \wr \ldots \wr R_{k_{s}}\), where \(s \geq 1\), each \(k_{i}\) is either prime or one of 4, 8, or 9, and \(k = k_{1}k_{2}\ldots k_{s}\).

5. A Bound on \(r(k,h)\)

Let \(d_{2} = d_{3} = 8\), and \(d_{h} = 4\) for all \(h > 3\).

Let \(b_{h} = \begin{cases} 1.45 & h = 2 \\ 1.49 & h = 3 \\ 1.48 & h > 3 \end{cases}\) and \(a_{h} = \left( \left( r(d_{h},h) + b_{h} \right)/24^{1/3} \right)^{1/d_{h}}\).

Let \(\overline{r}(k,h) = 24^{1/3}(a_{h})^{k} - b_{h}\), for all \(k, h \geq 2\). Then we have a lower bound on the number of orbits of imprimitive solvable groups as follows:

Theorem 3.6 \(\overline{r}(k,h) \leq r(k,h)\) for all \(k; h \geq 2\).
Proof: By definition \( r(k,h) = \frac{h}{k} \) for some \( k \) of the
form \( k = k_1 \ldots k_s \), where \( k = k_1 \ldots k_s \). We use
induction on \( s \).

Suppose first that \( s = 1 \), so \( k = R_k \). We consider the
possibilities for \( h \) and \( k \):

(i) \( h = 2 \) or \( 3 \). Then \( b_2 = 1.45 \), \( b_3 = 1.49 \), \( a_2 = 1.188 \)
and \( a_3 = 1.5064 \), and so by direct calculation we find that.
\( \tilde{r}(k,h) \leq r(k,h) \) for \( h = 2, 3 \) and \( k = 2, 3, 4, 5, 7, 8, 9 \) and 11.
Thus we need only consider \( k = R_q \), \( q \) prime, \( q > 11 \). But
\( \tilde{r}(q,h) = 24^{1/3} a_h q - b_h < h^q/(q(q-1)) < r(q,h) \) for \( q > 11 \).

(ii) \( h > 3 \), \( k = 4 \). Then we have
\( \tilde{r}(4,h) = 24^{1/3} [(r(4,h)+1.48)/24^{1/3}]^{1/4} = 1.48 = r(4,h) \).

(iii) \( h > 3 \), \( k = q \), \( q \) prime. Then using the exact
formula for \( r(4,h) \) from (2) we find that for \( h > 3 \),
\( r(4,h) + 1.48 < (h+1.5)^4/24 \). Thus we have
\( \tilde{r}(q,h) < 24^{1/3} [(h+1.5)^4/24^{4/3}]^{q/4} \). But \( r(q,h) > h^q/(q(q-1)) \).
Thus it is enough to show that
\[ 24^{1/3} q(q-1) < (24^{1/3} h/(h+1.5))^q \]
But this is true for \( h > 3 \) and \( q > 7 \), or \( h > 5 \) and \( q = 5 \),
or \( h > 13 \) and \( q = 3 \), or \( h > 7 \) and \( q = 2 \). Then testing
individually with the exact formulas the fifteen cases that
remain, we find that the result is true for all \( q > 2 \), \( h > 3 \).
(iv) $h > 3$, $k = 8$ or $9$. Then by Theorem 3.8 we need only consider $h = 4$, $k = 8$, or $h = 4, 5, 6$, $k = 9$. Testing these four cases individually gives us the result.

Thus we have $\bar{r}(k, h) \preceq r(k, h)$ for $s = 1$. Now suppose $s > 1$. Let $K' = R_{k_1} \text{ wr } \ldots \text{ wr } R_{k_{s-1}}, k' = k_1 k_2 \ldots k_{s-1}$, and $h' = (h) K'$, so $K = K' \text{ wr } R_{k_s}, (h) K = (h') R_{k_s}$, and we assume by induction that $\bar{r}(k', h) \preceq r(k', h)$. Then we have $r(k, h) = (h) K = (h') R_{k_s} = r(k_s, h')$, and so

$$\bar{r}(k, h) = 24^{1/3} a_h k - b_h$$

$$= 24^{1/3} \left[ (\bar{r}(k', h) + b_h) / 24^{1/3} \right]^k_s - b_h$$

$$\leq 24^{1/3} \left[ (r(k', h) + b_h) / 24^{1/3} \right]^k_s - b_h$$

$$\leq 24^{1/3} \left[ (h' + 1.49) / 24^{1/3} \right]^k_s - 1.45$$

Let $f(k_s, h') = 24^{1/3} \left[ (h' + 1.49) / 24^{1/3} \right]^k_s - 1.45$. Then we want $f(k_s, h') \leq r(k_s, h')$ so again we consider the possibilities for $k_s$:

(i) $k_s = 4$. Then $f(4, h') \leq r(4, h')$ if and only if

$$(h' + 1.49) / 24 - 1.45 \leq (h' + 6h'^3 + 11h'^2 + 6h') / 24,$$

and this is true for $h' > 60$. Thus we need only consider $h' \leq 60$. But there are only thirty-six combinations of $K'$ and $h$ that give $(h) K' \leq 60$ (listed on the next page) and considering the exact formulas in each of these cases we find that they all give $\bar{r}(k, h) \leq (h)(K' \text{ wr } R_4)$. 


(ii) \( k_s = q, q \) prime. Then to show \( f(q,h') \leq r(q,h') \) it is enough to show \( f(q,h') \leq (h')^q/(q(q-1)) \), or to show \( 24^{1/3} q(q-1) \leq (24^{1/3} h'/(h'+1.49))^q \). But this is true for \( h' \geq 3 \) and \( q \geq 11 \), or \( h' \geq 4 \) and \( q = 7 \), or \( h' \geq 6 \) and \( q = 5 \), or \( h' \geq 13 \) and \( q = 3 \), or \( h' \geq 8 \) and \( q = 2 \). Thus (from Table V) we need only consider the following possibilities for \( K', h \) and \( q \):

\[
\begin{array}{c|c|c}
  h & K' & q \\
  \hline
  2 & R_2 & 2,3,5,7 \\
  2 & R_3 & 2,3,5 \\
  2 & R_4 & 2,3,5 \\
  2 & R_5 & 2,3 \\
  2 & R_7 & 3 \\
  2 & R_2 \ wr \ R_3 & 3 \\
  2 & R_8 & 3 \\
  3 & R_2 & 2,3 \\
  3 & R_3 & 3 \\
  4 & R_2 & 3 \\
\end{array}
\]

Considering the exact formulas in each of these cases, we
find that they all give $\bar{r}(k, h) < (h)(K' \ wr \ R_q)$.

(iii) $k_s = 8$ or $9$. Then again we need only consider individually the cases where $k_s = 8$ and $h' = 3, 4$, or $k_s = 9$ and $h' = 3, 4, 5, 6$. As above, $\bar{r}(k, h) < (h)(K' \ wr \ R_{k_s})$ in each of these cases.

This completes the proof of the theorem.

6. A Limiting Case

Let $d_2 = d_3 = 8$ and $d_h = 4$ for $h > 3$ as before. Let $a'_h = [(r(d_h, h) + 1.5)/24]^{1/3}$. For $k > 1$, $h > 2$ let $\bar{r}(k, h) = 24^{1/3}a'_h^k - 1.5$. Then $\bar{r}(k, h)$ acts as an upper bound for a limiting case as follows:

Theorem 3.9 For any $N > 0$ and $h > 2$ there exists some $k > N$ such that $r(k, h) \leq \bar{r}(k, h)$.

Proof: Fix $h > 2$. Let $K_t = R_{d_h} \ wr \ R_4(t - 1)$. Let $k_t$ be the degree of $K_t$, so $k_t = 4^{t-1}d_h$. Then $r(k_t, h) < (h)K_t$, and we will prove by induction on $t$ that $(h)K_t \leq \bar{r}(k_t, h)$.

If $t = 1$, then $(h)K_1 = (h)R_{d_h} = r(d_h, h)$, and $\bar{r}(k_1, h) = \bar{r}(d_h, h) = 24^{1/3}(a'_h)^{d_h} - 1.5 = r(d_h, h)$ by the definition of $a'_h$. Thus $(h)K_1 \leq \bar{r}(k_1, h)$. 

Now suppose \( t > 1 \). Let \( h' = (h)K_{t-1} \) and 
\[
\tilde{h}' = \tilde{r}(k_{t-1}, h),
\]
and we assume \( h' \leq \tilde{h}' \). Note that since \( t > 1 \), \( (h)K_{t-1} > (2)R_4 = 5 \), and so \( \tilde{h}' > 4 \). Thus 
\[
(h)K_t = (h)K_{t-1} \wr R_4
\]
\[
= r(4, h') 
\]
\[
\leq r(4, \tilde{h}') 
\]
\[
= (\tilde{h}'^4 + 6\tilde{h}'^3 + 11\tilde{h}'^2 + 6\tilde{h}')/24 
\]
\[
< (\tilde{h}' + 1.5)^4/24 - 1.5 
\]
\[
= [(24^{1/3}a_{h_{t-1}}(k_{t-1}) - 1.5) + 1.5]^4/24 - 1.5 
\]
\[
= 24^{1/3}a_{h_{t-1}}(k_{t-1}) - 1.5 
\]
\[
= \tilde{r}(k_{t-1}, h) 
\]
Thus the result is true for all \( t > 1 \) and \( h > 2 \), and so the theorem is proved.

Thus by combining Theorem 3.8 and Theorem 3.9 we have that for any \( h > 2 \) and \( N > 0 \) there exists a transitive solvable subgroup \( K \) of \( S_k \), where \( k > N \), such that
\[
24^{1/3}a_h^k - 1.5 \leq (h)K \leq 24^{1/3}a_h^k - 1.5 
\]
Table VI gives a comparison of the values of \( a_h \) and \( a_h' \) for small \( h \).
Table VI. Values of $a_h$ and $a'_h$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$a_h$</th>
<th>$a'_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.188069</td>
<td>1.188716</td>
</tr>
<tr>
<td>3</td>
<td>1.506405</td>
<td>1.506429</td>
</tr>
<tr>
<td>4</td>
<td>1.885414</td>
<td>1.886060</td>
</tr>
<tr>
<td>5</td>
<td>2.231148</td>
<td>2.231304</td>
</tr>
<tr>
<td>6</td>
<td>2.578356</td>
<td>2.578457</td>
</tr>
<tr>
<td>7</td>
<td>2.926171</td>
<td>2.926240</td>
</tr>
<tr>
<td>8</td>
<td>3.274136</td>
<td>3.274185</td>
</tr>
<tr>
<td>9</td>
<td>3.622078</td>
<td>3.622114</td>
</tr>
<tr>
<td>10</td>
<td>3.969935</td>
<td>3.969962</td>
</tr>
</tbody>
</table>

Note: The values of $a_h$ and $a'_h$ can be improved by taking higher values of $d_h$ (which means the values of $b_h$ will also increase, approaching 1.5). For example, taking $d_2 = 32$ gives $a_2 = [\left(\frac{r(32,2)+1.499}{24^{1/3}}\right)^{1/32} = 1.188087117$. and $a'_2 = [\left(\frac{r(32,2)+1.5}{24^{1/3}}\right)^{1/32} = 1.188087169$.

Theorem 3.8 is still true when this $a_2$ is used, but the number of exceptional cases that must be checked in the proof is much larger. The values of $d_h$ used must be of the form $8 \cdot 4^t$ when $h = 2, 3$ and $4 \cdot 4^t$ when $h > 3$, for some $t \geq 0$, and it is possible that if large $d_h$ are used in Theorem 3.8 there may be some exceptions involving $k < d_h$. 
7. An Exact Formula for $r(k,h)$

Define $\lambda(k,h) = t$, where $r(k,h) = (h)(R_t \wr K')$, $K'$ a transitive solvable subgroup of $S_k$, with $k = k't$, and $t$ maximal such that $r(k,h)$ can be written in this form. So for example $\lambda(6,3) = 3$, since $(3)(R_3 \wr R_2) = 55$ and $(3)(R_2 \wr R_3) = 56$, and these are the only two maximal transitive solvable subgroups of $S_6$.

Now given $h,k \geq 2$, let $h_0 = h$, $k_0 = k$, and $t_0 = \lambda(k_0,h_0)$. Let $k_{i+1} = k_i/t_i$, $h_{i+1} = r(t_i,h_i)$, and $t_{i+1} = \lambda(k_{i+1},h_{i+1})$, $i=0,1,...,s-1$ where $s$ is such that $k_s = 1$. Then by the definition of $\lambda(k,h)$, $r(k,h) = h_s$. Thus $\lambda(k,h)$ can be used to calculate $r(k,h)$ exactly for any $k,h$.

So let $q_i$ be the $(i)$th prime, for all $i \geq 1$, and for any $k \geq 1$ let $s(k,i) = \frac{a_1}{q_1} ... \frac{a_{i-1}}{q_{i-1}}$ where $k = \frac{a_1}{q_1} ... \frac{a_m}{q_m}$ is the prime decomposition of $k$. Then we can calculate $\lambda(k,h)$ as follows:

**Theorem 3.8** For any $k \geq 2$, $h \geq 2$ we have

$$\lambda(k,h) = \begin{cases} 4 & \text{if } 4 \mid k \\ 3 & \text{else if } 3 \mid k \\ 5 & \text{else if } 5 \mid k \\ 2 & \text{else if } 2 \mid k \\ q & \text{else if } q \text{ is the least prime with } q \mid k \end{cases}$$

with the following exceptions:
\[
\lambda(k, 2) = \left\{ \begin{array}{l}
11 \text{ if } 11 \mid k, s(k, 11) \mid 9, 10, 12, 14, \text{ or } 16 \\
7 \text{ else if } 7 \mid k, s(k, 7) \mid 16 \text{ or } 90, \\
\quad \text{ or } 4 \mid k, 8 \nmid k \\
5 \text{ else if } 5 \mid k, s(k, 5) \mid 12 \\
\quad \text{ or } 16 \mid k \text{ and } 9, 32 \nmid k \\
9 \text{ else if } 9 \mid k, s(k, 3) = 1, 2, 4, \text{ or } 16 \\
8 \text{ else if } 8 \mid k \\
13 \text{ else if } 13 \mid k, s(k, 13) \mid 12 \\
17 \text{ else if } 17 \mid k, s(k, 17) \mid 6 \\
19 \text{ else if } 19 \mid k, s(k, 19) \mid 2
\end{array} \right.
\]

\[
\lambda(k, 3) = \left\{ \begin{array}{l}
7 \text{ if } 7 \mid k, s(k, 7) = 4 \cdot 3 \cdot 5^a \text{ or } 4 \cdot 5^a \text{ (} a > 0 \text{)} \\
\quad \text{ or } s(k, 7) \mid 6 \\
5 \text{ else if } 5 \mid k, s(k, 5) = 16 \text{ or } s(k, 5) \mid 12 \\
8 \text{ else if } 8 \mid k \\
9 \text{ else if } 9 \mid k \\
11 \text{ else if } 11 \mid k, s(k, 11) \mid 2 \text{ or } 3 \\
13 \text{ else if } 13 \mid k, s(k, 13) \mid 6 \\
9 \text{ if } 9 \mid k, 32 \nmid k \\
5 \text{ else if } 5 \mid k, s(k, 5) \mid 24 \\
8 \text{ else if } 8 \mid k, k \equiv 2 \pmod{4}, \text{ and } 128, 96 \nmid k \\
7 \text{ else if } 7 \mid k, s(k, 7) \mid 6 \\
11 \text{ else if } 11 \mid k, s(k, 11) \mid 2 \\
\end{array} \right.
\]

\[
\lambda(k, 4) = \left\{ \begin{array}{l}
9 \text{ if } 9 \nmid k, 4 \nmid k \\
5 \text{ else if } 5 \mid k, s(k, 5) \mid 6, 8 \\
7 \text{ else if } 7 \mid k, s(k, 7) \mid 6 \\
\end{array} \right.
\]

\[
\lambda(k, 5) = \left\{ \begin{array}{l}
5 \text{ if } 5 \mid k, s(k, 5) \mid 18 \\
9 \text{ else if } 9 \mid k, 4 \nmid k \\
7 \text{ else if } 7 \mid k, s(k, 7) \mid 2 \\
\end{array} \right.
\]
\[ \lambda(k, 7) = \begin{cases} \\
5 & \text{if } 5 \mid k, s(k, 5) \mid 18 \\
7 & \text{else if } 7 \mid k \\
\end{cases} \]

\[ \lambda(k, h) = \begin{cases} \\
5 & \text{if } 5 \mid k, s(k, 5) \mid 6 \\
7 & \text{else if } 7 \mid k, s(k, 7) \mid 2 \\
\quad \text{for } h = 8, 9, 10 \\
\end{cases} \]

\[ \lambda(k, h) = 7 \quad \text{if } 7 \mid k, s(k, 7) \mid 2 \\
\quad \text{for } h = 11, 12, 13, 14, 15 \]

Note: Apart from the exceptions, using this theorem to produce \( K \) such that \( r(k, h) = (h)K \) results in \( K \) having maximal order as a transitive solvable subgroup of \( S_k \)

([10] Theorem 2).

Proof: If \( k \) is prime, or \( k = 4 \), or \( k = 8 \) and \( h = 2, 3, 4 \) or \( k = 9 \) and \( h = 2, 3, 4, 5, 6 \) then \( r(k, h) = (h)R_k \) and so \( \lambda(k, h) = k \). So suppose now that \( k = t_1t_2 \), where \( t_1 \) is either prime, or 4, or 8 with \( h = 2, 3, 4 \) or 9 with \( h = 2, 3, 4, 5, 6 \) and \( t_1 \neq t_2 \). Then to find \( \lambda(k, h) \) we must compare \( (h)(R_{t_1} \wr R_{t_2}) \) and \( (h)(R_{t_2} \wr R_{t_1}) \).

Suppose first that \( t_1 = 3 \) and \( t_2 = q \), \( q \) prime, \( q > 3 \). Then \( r(3, h) = (h^3 + 3h^2 + 2h)/6 \) and so

\[ (h + 0.8)^3/6 < (h)R_3 < (h + 1)^3/6 \quad \text{for all } h \leq 2. \]

Similarly,

\[ h^q/(q(q-1)) < (h)R_q < h^q/(q(q-1)) + h^{(q+1)/2} \]
and thus we have

\[(h)(R_q \text{ wr } R_q) < \left(\frac{(h+1)^3}{6}\right)^q / (q(q-1)) + \left(\frac{(h+1)^3}{6}\right)^{(q+1)/2}\]

and

\[(h)(R_q \text{ wr } R_3) > \left(\frac{h^q}{q(q-1)} + 0.8\right)^3 / 6\]

and so we want

\[\left(\frac{(h+1)^3}{6}\right)^q / (q(q-1)) \left[1 + \frac{q(q-1)}{(h+1)^3/6} \right]^{(q-1)/2}\]

\[< \left(\frac{h^q}{q(q-1)} + 0.8\right)^3 / 6\]

Taking logarithms, it is enough if

\[q \ln\left(\frac{(h+1)^3}{6}\right) - \ln[q(q-1)] + q(q-1) / \left[\left(\frac{(h+1)^3}{6}\right)^{(q-1)/2}\right]\]

\[< 3 \ln\left[\frac{h^q}{q(q-1)} + 0.8\right] - \ln 6\]

or

\[3q \ln(h+1) - (q-1)\ln 6 - \ln[q(q-1)] + q(q-1) / \left[4.5(q-1)^2\right]\]

\[< 3q \ln(h) - 3 \ln[q(q-1)]\]

or

\[3q[\ln(h+1) - \ln(h) - (\ln 6)/3] + \ln 6 + 2\ln[q(q-1)] + q(q-1) / \left[4.5(q-1)^2\right] < 0\]

Now \(\ln(h+1) - \ln(h) - (\ln 6)/3 < -0.19\) for \(h > 2\). Thus we want

\[-0.57 q + \ln 6 + 2\ln[q(q-1)] + q(q-1) / \left[4.5(q-1)^2\right] < 0\]
and this is true for \( q \geq 29 \). Similarly, by considering

\( h > 2 \) we find that the result is true for \( h \geq 3 \) and \( q \geq 17 \),

for \( h \geq 4 \) and \( q \geq 11 \), for \( h \geq 6 \) and \( q \geq 7 \), and for \( h \geq 16 \)

and \( q \geq 5 \). But this leaves twenty-five possibilities for

\( q \) and \( h \) which we can check individually using the exact

formulas, and we find that \((h)(R_3 \text{ wr } R_q) < (h)(R_q \text{ wr } R_3)\)

except for the following cases:

\[(h)(R_5 \text{ wr } R_3) < (h)(R_3 \text{ wr } R_5), h = 2, 3, \ldots, 10\]

\[(h)(R_7 \text{ wr } R_3) < (h)(R_3 \text{ wr } R_7), h = 2, 3, 4, 5\]

\[(h)(R_{11} \text{ wr } R_3) < (h)(R_3 \text{ wr } R_{11}), h = 2, 3\]

\[(2)(R_{13} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{13})\]

\[(2)(R_{17} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{17})\]

\[(2)(R_{19} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{19})\]

By similarly considering every other combination of

\( t_1 \) and \( t_2 \) we find that

\[(h)(R_4 \text{ wr } R_3) \leq (h)(R_3 \text{ wr } R_4)\]

\[(h)(R_4 \text{ wr } R_2) \leq (h)(R_2 \text{ wr } R_4)\]

\[(h)(R_3 \text{ wr } R_2) \leq (h)(R_2 \text{ wr } R_3)\] \quad \text{for all } h \geq 2 \quad (5)

\[(h)(R_5 \text{ wr } R_2) \leq (h)(R_2 \text{ wr } R_5)\]

and otherwise

\[(h)(R_{t_1} \text{ wr } R_{t_2}) < (h)(R_{t_2} \text{ wr } R_{t_1}) \quad \text{whenever } t_1 < t_2\]

with the following exceptions:
(h) \((R_8 \text{ wr } R_t) < (h)(R_t \text{ wr } R_8)\ h = 2, 3, 4, \text{ for all } t\)
(h) \((R_9 \text{ wr } R_t) < (h)(R_t \text{ wr } R_9)\ h = 2, 3, 4, 5, 6, t \neq 8\)
(2) \((R_{11} \text{ wr } R_7) < (2)(R_7 \text{ wr } R_{11})\)
(2) \((R_7 \text{ wr } R_5) < (2)(R_5 \text{ wr } R_7)\)
(2) \((R_{11} \text{ wr } R_5) < (2)(R_{11} \text{ wr } R_5)\)
(h) \((R_5 \text{ wr } R_3) < (h)(R_3 \text{ wr } R_5)\ h = 2, 3, \ldots, 10\)
(h) \((R_7 \text{ wr } R_3) < (h)(R_3 \text{ wr } R_7)\ h = 2, 3, 4, 5\)
(h) \((R_{11} \text{ wr } R_3) < (h)(R_3 \text{ wr } R_{11})\ h = 2, 3\)
(2) \((R_{13} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{13})\)
(h) \((R_5 \text{ wr } R_4) < (h)(R_4 \text{ wr } R_5)\ h = 2, 3, 4, 5\)
(h) \((R_7 \text{ wr } R_4) < (h)(R_4 \text{ wr } R_7)\ h = 2, 3\)
(2) \((R_{11} \text{ wr } R_4) < (2)(R_4 \text{ wr } R_{11})\)
(2) \((R_{13} \text{ wr } R_4) < (2)(R_4 \text{ wr } R_{13})\)
(h) \((R_7 \text{ wr } R_2) < (h)(R_2 \text{ wr } R_7)\ h = 2, 3, \ldots, 16\)
(h) \((R_{11} \text{ wr } R_2) < (h)(R_2 \text{ wr } R_{11})\ h = 2, 3, 4\)
(h) \((R_{13} \text{ wr } R_2) < (h)(R_2 \text{ wr } R_{13})\ h = 2, 3\)
(2) \((R_{17} \text{ wr } R_2) < (2)(R_2 \text{ wr } R_{17})\)
(2) \((R_{19} \text{ wr } R_2) < (2)(R_2 \text{ wr } R_{19})\)
(2) \((R_{23} \text{ wr } R_2) < (2)(R_{23} \text{ wr } R_2)\)
(2) \((R_{17} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{17})\)
(2) \((R_{19} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{19})\)

Since none of these exceptions include primes greater than 23, it follows that for any prime \(q > 23\), \(\lambda(k, h) = q\) if and only if \(q \mid k\) and \(t \nmid k\) for any \(t < q\). So we consider now under which circumstances \(\lambda(k, h) = t\) although \(q \mid k, q < t\).
Suppose first that \( t = 5 \). Then we want to know when
\[ \lambda(k,h) = 5 \text{ although } k \text{ is divisible by some prime less than } 5, \]
and so we may assume \( k = 2^a 3^b 5 \), for some \( a, b \) such that \( a + b > 0 \). Let \( d = a/2 \) if \( a \) is even, and \( (a-1)/2 \) if \( a \) is odd, and let \( c = a - 2d \). Then by (5) we know that
\[ r(2^a 3^b, h) = (h) (R_4^{(d)} \ wr R_3^{(b)} \ wr R_2^{(c)}), \]
except possibly when \( h = 2, 3, 4 \) and we may have \( (h) (R_8 \ wr K') \) for some \( K' \), or \( h = 2, 3, 4, 5, 6 \) and we may have \( (h) (R_9 \ wr K') \) for some \( K' \).

Thus we wish to know when
\[ r(k, h) = (h) (R_5 \ wr R_4^{(d)} \ wr R_3^{(b)} \ wr R_2^{(c)}), \] (6)
or possibly if \( h = 2 \),
\[ r(k, 2) = (2) (R_5 \ wr R_9 \ wr R_4^{(d)} \ wr R_3^{(b-2)} \ wr R_2^{(c)}). \] (7)

We consider the possibilities for \( d \) and \( b \):

(i) \( d > 1 \). Then we know that the only possibilities for \( (h) (R_5 \ wr R_4) < (h) (R_4 \ wr R_5) \) are \( h = 2, 3, 4, 5, \) and so
\[ (h) (R_4 \ wr R_5 \ wr R_4^{(d-1)} \ wr R_3^{(b)} \ wr R_2^{(c)}), \]
\[ < (h) (R_5 \ wr R_4^{(d)} \ wr R_3^{(b)} \ wr R_2^{(c)}), \]
for all \( h > 5 \), and thus (6) can only be true for \( h = 2, 3, 4, 5 \).

But \( (h) (R_4^{(2)} \ wr R_5) < (h) (R_5 \ wr R_4^{(2)}) \) for \( h = 4, 5 \), and so as above we need only consider \( h = 2, 3 \). But for \( h = 2, 3 \),
\[ (h) (R_5 \ wr R_4^{(2)}) < (h) (R_8 \ wr R_5 \ wr R_2) < (h) (R_4^{(2)} \ wr R_5), \]
and so (6) is true for $k = 2^45$. But

\[(h)(R_8 \text{ wr } R_4 \text{ wr } R_5) < (h)(R_5 \text{ wr } R_4^{(2)} \text{ wr } R_2), \ h = 2, 3\]

and so (6) is not true for $k = 2^55$. Further

\[(h)(R_8 \text{ wr } R_4 \text{ wr } R_5 \text{ wr } R_2) < (h)(R_5 \text{ wr } R_4^{(3)}), \ h = 2, 3\]

and so as above (6) is not true for $d > 2$. Also

\[(3)(R_8 \text{ wr } R_3 \text{ wr } R_5 \text{ wr } R_2) < (3)(R_5 \text{ wr } R_4^{(2)} \text{ wr } R_3)\]

and so (6) is not true for $h = 3$, $d = 2$, and $b > 0$. Thus the only remaining possibility is $h = 2$, $d = 2$. But

\[(2)(R_8 \text{ wr } R_4 \text{ wr } R_3 \text{ wr } R_5) < (2)(R_5 \text{ wr } R_4^{(2)} \text{ wr } R_3 \text{ wr } R_2)\]

and

\[(2)(R_8 \text{ wr } R_3^{(2)} \text{ wr } R_5 \text{ wr } R_2) < (2)(R_5 \text{ wr } R_4^{(2)} \text{ wr } R_3^{(2)})\]

and so (6) is not true for $h = 2$, $d = 2$ unless $c + b = 0$; i.e. $k = 2^45$.

(ii) $d = 1$. As above we may assume $h < 6$. Then since

\[(h)(R_5 \text{ wr } R_4) < (h)(R_5 \text{ wr } R_5) \text{ for } h = 2, 3, 4, 5, \ (6) \text{ is true}\]

for $k = 2^25$. Also, \((h)(R_5 \text{ wr } R_4 \text{ wr } R_3) < (h)(R_4 \text{ wr } R_3 \text{ wr } R_5)\)

for $h = 2, 3, 4$, and so (6) is true for $k = 2^23^15$, $h = 2, 3, 4$, but not for $h = 5$. \((5)(R_5 \text{ wr } R_4 \text{ wr } R_3) < (5)(R_4 \text{ wr } R_5 \text{ wr } R_2)\)

and \((4)(R_5 \text{ wr } R_4 \text{ wr } R_2) < (4)(R_8 \text{ wr } R_5)\), but for $h = 2, 3$,

\[(h)(R_8 \text{ wr } R_5) < (h)(R_5 \text{ wr } R_4 \text{ wr } R_2) \text{ and so (6) is true for}\]

\[\]
$k = 2^3 5$ when $h = 4, 5$ but not when $h = 2, 3$. Thus the only possibility left to consider is $b > 1$. But since

$$(5)(R_4 \ wr R_3 \ wr R_5) < (5)(R_5 \ wr R_4 \ wr R_3), \ (6) \text{ is not true for } h = 5, \ b > 0. \ \text{Also,}$$

$$(h)(R_9 \ wr R_4 \ wr R_5) < (h)(R_5 \ wr R_4 \ wr R_3^{(2)})$$

for $h = 2, 3, 4, 5$ and so $(6)$ is not true for $b > 1$.

(iii) $d = 0, \ b > 0$. Then we know that

$$(h)(R_5 \ wr R_3) < (h)(R_3 \ wr R_5) \text{ if and only if } h < 11, \ \text{and so} \ (6) \text{ is true for } b = 1 \text{ and } h < 11, \ \text{and we may assume now } h < 11. \ \text{Now,} \ (6)(R_5 \ wr R_3^{(2)}) < (6)(R_9 \ wr R_5) \text{ but}$$

$$(h)(R_9 \ wr R_5) < (h)(R_5 \ wr R_3^{(2)}) \text{ for } h = 2, 3, 4, 5, \ \text{and thus}$$

$$(h)(R_9 \ wr R_5 \ wr R_3^{(b-2)} \ wr R_2^{(c)}) < (h)(R_5 \ wr R_3^{(b)} \ wr R_2^{(c)})$$

for all $b > 1, \ h = 2, 3, 4, 5$, and $(6)$ is not true in these cases. So we need now only consider $h = 6, 7, 8, 9, 10$. But

$$(h)(R_3^{(2)} \ wr R_9) < (h)(R_5 \ wr R_3^{(2)}) \text{ for } h = 8, 9, 10, \ \text{and so we need only consider } h = 6, 7. \ \text{We have}$$

$$(7)(R_5 \ wr R_3^{(2)}) < (7)(R_3^{(2)} \ wr R_5), \ \text{and so} \ (6) \text{ is true for } k = 3^2 5 \text{ and } h = 6, 7. \ \text{Also for } k = 2^1 3^2 5 \text{ we have}$$

$$(6)(R_5 \ wr R_3^{(2)} \ wr R_2) < (6)(R_9 \ wr R_5 \ wr R_2) \ \text{and}$$

$$(7)(R_5 \ wr R_3^{(2)} \ wr R_2) < (7)(R_3^{(2)} \ wr R_5 \ wr R_2), \ \text{and so} \ (6) \text{ is true. But for } b > 2 \text{ we have}$$
(6) \((R_9 \text{ wr } R_3 \text{ wr } R_5) < (6)(R_5 \text{ wr } R_3^{(3)})\) and
\[(7)(R_3^{(3)} \text{ wr } R_5) < (7)(R_5 \text{ wr } R_3^{(3)})\], and so (6) is not true for \(b > 2\).

(iv) \(d = 0, b = 0\). But then the only possibility is \(c = 1\), and we know \((h)(R_5 \text{ wr } R_2) < (h)(R_2 \text{ wr } R_5)\) for all \(h\). Thus (6) is always true in this case.

By a similar investigation we find that no new possibilities for \(k\) are added by (7). Thus by collecting all the cases above when (6) is true, we find that \(\lambda(k,h) = 5\) for \(k = 2^3 3^b 5\) only as follows:

<table>
<thead>
<tr>
<th>k</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2^1 3^5)</td>
<td>any (h)</td>
</tr>
<tr>
<td>(2^2 3^5)</td>
<td>2,3,4,5</td>
</tr>
<tr>
<td>(2^4 3^5)</td>
<td>2,3</td>
</tr>
<tr>
<td>(2^4 3^1 5)</td>
<td>2</td>
</tr>
<tr>
<td>(2^2 3^1 5)</td>
<td>2,3,4</td>
</tr>
<tr>
<td>(2^3 5)</td>
<td>4,5</td>
</tr>
<tr>
<td>(2^3 3^1 5)</td>
<td>4</td>
</tr>
<tr>
<td>(2^1 3^1 5)</td>
<td>2,3,\ldots,10</td>
</tr>
<tr>
<td>(3^2 5)</td>
<td>6,7</td>
</tr>
<tr>
<td>(2^1 3^2 5)</td>
<td>6,7</td>
</tr>
</tbody>
</table>

Similar analyses for \(t = 2,3,4,7,8,9,11,13,17,19,23\) complete the proof of the theorem.
Table VII. Values of $r(k,h)$ less than 1000

<table>
<thead>
<tr>
<th>h</th>
<th>k</th>
<th>$r(k,h)$</th>
<th>h</th>
<th>k</th>
<th>$r(k,h)$</th>
<th>h</th>
<th>k</th>
<th>$r(k,h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>7</td>
<td>78</td>
<td>29</td>
<td>2</td>
<td>435</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>84</td>
<td>2</td>
<td>3</td>
<td>455</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>2</td>
<td>91</td>
<td>6</td>
<td>5</td>
<td>462</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>6</td>
<td>14</td>
<td>2</td>
<td>105</td>
<td>2</td>
<td>25</td>
<td>462</td>
</tr>
<tr>
<td>2</td>
<td>.7</td>
<td>10</td>
<td>10</td>
<td>2</td>
<td>110</td>
<td>30</td>
<td>2</td>
<td>465</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>10</td>
<td>8</td>
<td>3</td>
<td>120</td>
<td>2</td>
<td>22</td>
<td>465</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>10</td>
<td>15</td>
<td>2</td>
<td>120</td>
<td>9</td>
<td>4</td>
<td>495</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>10</td>
<td>6</td>
<td>4</td>
<td>126</td>
<td>31</td>
<td>2</td>
<td>496</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>10</td>
<td>2</td>
<td>20</td>
<td>126</td>
<td>4</td>
<td>8</td>
<td>496</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>14</td>
<td>16</td>
<td>2</td>
<td>136</td>
<td>2</td>
<td>17</td>
<td>522</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>15</td>
<td>3</td>
<td>9</td>
<td>140</td>
<td>32</td>
<td>2</td>
<td>528</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>17</td>
<td>2</td>
<td>153</td>
<td>14</td>
<td>3</td>
<td>560</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>20</td>
<td>9</td>
<td>3</td>
<td>165</td>
<td>2</td>
<td>27</td>
<td>560</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>21</td>
<td>18</td>
<td>2</td>
<td>171</td>
<td>33</td>
<td>2</td>
<td>561</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>21</td>
<td>19</td>
<td>2</td>
<td>190</td>
<td>34</td>
<td>2</td>
<td>595</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>24</td>
<td>5</td>
<td>5</td>
<td>201</td>
<td>35</td>
<td>2</td>
<td>630</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>28</td>
<td>20</td>
<td>2</td>
<td>210</td>
<td>5</td>
<td>6</td>
<td>630</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>80</td>
<td>4</td>
<td>6</td>
<td>210</td>
<td>36</td>
<td>2</td>
<td>666</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>35</td>
<td>7</td>
<td>4</td>
<td>210</td>
<td>15</td>
<td>3</td>
<td>680</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>35</td>
<td>10</td>
<td>3</td>
<td>220</td>
<td>3</td>
<td>12</td>
<td>680</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>35</td>
<td>2</td>
<td>21</td>
<td>220</td>
<td>37</td>
<td>2</td>
<td>703</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>36</td>
<td>2</td>
<td>24</td>
<td>220</td>
<td>10</td>
<td>4</td>
<td>715</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>45</td>
<td>21</td>
<td>2</td>
<td>231</td>
<td>2</td>
<td>28</td>
<td>715</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>55</td>
<td>22</td>
<td>2</td>
<td>253</td>
<td>2</td>
<td>32</td>
<td>715</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>55</td>
<td>23</td>
<td>2</td>
<td>276</td>
<td>38</td>
<td>2</td>
<td>741</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>55</td>
<td>11</td>
<td>3</td>
<td>286</td>
<td>39</td>
<td>2</td>
<td>780</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>55</td>
<td>24</td>
<td>2</td>
<td>300</td>
<td>16</td>
<td>3</td>
<td>816</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>56</td>
<td>3</td>
<td>10</td>
<td>300</td>
<td>40</td>
<td>2</td>
<td>820</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>56</td>
<td>25</td>
<td>2</td>
<td>325</td>
<td>41</td>
<td>2</td>
<td>861</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>66</td>
<td>8</td>
<td>4</td>
<td>330</td>
<td>42</td>
<td>2</td>
<td>903</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>70</td>
<td>26</td>
<td>2</td>
<td>351</td>
<td>43</td>
<td>2</td>
<td>946</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>74</td>
<td>12</td>
<td>3</td>
<td>364</td>
<td>7</td>
<td>5</td>
<td>952</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>75</td>
<td>27</td>
<td>2</td>
<td>378</td>
<td>17</td>
<td>3</td>
<td>969</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>76</td>
<td>28</td>
<td>2</td>
<td>406</td>
<td>44</td>
<td>2</td>
<td>990</td>
</tr>
</tbody>
</table>
Chapter IV. The General Bound

1. The Product Action of the Wreath Product

We consider now another form of the wreath product of permutation groups, which is analogous to the action of the wreath product on a linear space. Let \( K, N \) be subgroups of \( \text{Sym}(X), \text{Sym}(Y) \) respectively; where \( |X| = k, |Y| = n \).

Let \( G = \{ (r_1, \ldots, r_n; s) | r_i \in K, s \in N \} \). For the element \( r = (r_1, \ldots, r_n; s) \) of \( G \) define an action of \( r \) on \( X^n \) by

\[
r(x_1, \ldots, x_n) = (r_{-1}(x_1), \ldots, r_{-1}(x_n)).
\]

Then for any \( t = (t_1, \ldots, t_n; s') \in G \) we have

\[
rt = (r_{s'}(1)t_1, \ldots, r_{s'}(n)t_n; ss') \in G,
\]

and so \( G \) is a subgroup of \( \text{Sym}(X^n) \), and is isomorphic as an abstract group to the wreath product \( K \wr N \) in its imprimitive action on \( X \times Y \). \( G \) is called the wreath product \( K \wr N \) in its product action on \( X^n \).

Theorem 4.1 Let \( N \) be a transitive subgroup of \( S_n \), and \( K \) a primitive subgroup of \( \text{Sym}(X) \), \( K \) not regular of prime order. Then \( K \wr N \) is primitive in its product action on \( X^n \).


The relationship between the product action of the wreath product and the wreath product of a linear group and a permutation group can be seen in the following theorem:
Theorem 4.2 Let $\overline{G}$ be a primitive solvable permutation group acting on the set $V$, with stabilizer $G$ a subgroup of the linear group $GL(V)$, so $G = V \cdot G$. Suppose $G$ has the form $G = H \wr K$, where $K$ is a transitive solvable subgroup of $S_k$, and $H$ is a primitive subgroup of $GL(W)$ for some space $W$ such that $V = W^k$. Let $\overline{H}$ be the primitive permutation group $W \cdot H$ (as in Theorem 1.2). Then $\overline{G}$ is permutation isomorphic to the wreath product $\overline{H} \wr K$ in its product action on $W^k$.

Proof: Let $\overline{g} \in \overline{G}$. Then $\overline{g} = vg$ for some $v \in V$, $g \in G$, and for any $x \in V$, $\overline{g}(x) = vg(x) = g(x) + v$. Since $G = H \wr K$, $g = (h_1, \ldots, h_k; r)$ for some $h_i \in H$, $r \in K$. Since $V = W^k$, $v = (w_1, \ldots, w_k)$ for some $w_i \in W$. So define $\psi: G \to H \wr K$ by

$$\psi(\overline{g}) = (w_{\overline{r}(1)} h_1, \ldots, w_{\overline{r}(k)} h_k; r),$$

where $w_{\overline{r}(i)} h_i$ is in $\overline{H}$ since $H = W \cdot H$. Then for any $x \in V$,

$$x = (x_1, \ldots, x_k) \in W^k,$$

we have (for $x_i = x_{\overline{r}^{-1}(i)}$),

$$\psi(\overline{g})(x) = (w_{\overline{r}(1)} h_1, \ldots, w_{\overline{r}(k)} h_k; r)(x_1, \ldots, x_k)$$

$$= (w_1 h_{\overline{r}^{-1}(1)} x'_1, \ldots, w_k h_{\overline{r}^{-1}(k)} x'_k)$$

$$= (h_{\overline{r}^{-1}(1)} (x'_1) + w_1, \ldots, h_{\overline{r}^{-1}(k)} (x'_k) + w_k)$$

$$= (h_1, \ldots, h_k; r)(x_1, \ldots, x_k) + (w_1, \ldots, w_k)$$

$$= g(x) + v$$

$$= \overline{g}(x).$$

Thus $\overline{G}$ and $H \wr K$ are permutation isomorphic in $\text{Sym}(W^k)$.
2. Degrees with Small Rank

Now let $\overline{G}$ be a maximal solvable primitive permutation group of degree $d = p^n$ and rank $r$. We look at the cases in which $r$ can be at most 10.

Let $G$ be the stabilizer of $\overline{G}$, so $G$ is a maximal solvable irreducible subgroup of $\text{GL}(n,p)$. Suppose first that $G$ has a semilinear component, so $G = (T(p^m) \wr K)$ where $K$ is a maximal solvable transitive subgroup of $S_k$, $k \geq 1$, and $n = km$. Then by Theorem 4.2 $\overline{G} = (S(p^m) \wr K)$. Further the rank of $\overline{G}$ is the number of orbits of $G$, so $r = 2$ if $k = 1$ and $r \geq r(k,2)$ if $k > 1$. Thus by using Table VII we can list all such groups with rank at most 10:

Table VIII. Semilinear-Component Groups

<table>
<thead>
<tr>
<th>Group</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(p^m)$</td>
<td>2</td>
</tr>
<tr>
<td>$S(p^m) \wr R_2$</td>
<td>3</td>
</tr>
<tr>
<td>$S(p^m) \wr R_3$</td>
<td>4</td>
</tr>
<tr>
<td>$S(p^m) \wr R_4$</td>
<td>5</td>
</tr>
<tr>
<td>$S(p^m) \wr R_5$</td>
<td>6</td>
</tr>
<tr>
<td>$S(p^m) \wr R_7$</td>
<td>10</td>
</tr>
<tr>
<td>$S(p^m) \wr R_8$</td>
<td>10</td>
</tr>
<tr>
<td>$S(p^m) \wr R_2 \wr R_3$</td>
<td>10</td>
</tr>
<tr>
<td>$S(p^m) \wr R_3 \wr R_2$</td>
<td>10</td>
</tr>
</tbody>
</table>
Now suppose $G$ has no semilinear component, and $\mathcal{C}$ has rank $r$. Then $G = H \wr K$, where $H$ is a maximal solvable primitive non-semilinear subgroup of $GL(p^m)$, $K$ is a maximal solvable transitive subgroup of $S_k$, $k \geq 1$ and $n = km$. Let $h$ be the number of orbits of $H$, so $h \geq (r^s(p^m) + 1)$. Then $r = h$ if $k = 1$ and $r = (h)K > r(k,h)$ if $k > 1$. Thus from Table III and Table VII we can compute the least possible rank of $\mathcal{C}$ given its degree $p^n$ as follows:

Table IX. Non-Semilinear-Component Groups

<table>
<thead>
<tr>
<th>Rank</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$3^2, 5^2, 7^2, 11^2, 23^2$</td>
</tr>
<tr>
<td>3</td>
<td>$13^2, 17^2, 19^2, 29^2, 31^2, 47^2, 3^4, 5^4, 7^4, 11^4, 23^4, 2^6, 3^6$</td>
</tr>
<tr>
<td>4</td>
<td>$37^2, 41^2, 43^2, 53^2, 59^2, 71^2, 7^3, 5^6, 7^6, 11^6, 23^6, 3^10$</td>
</tr>
</tbody>
</table>
| 5    | $61^2, 67^2, 79^2, 83^2, 3^8, 5^8, 11^8, 23^8$  
$(13^2, 12^2, 15^2, 16^2, 18^2)^+$ |
| 6    | $97^2, 109^2, 113^2, 127^2, 131^2, 13^3, 17^4, 19^4, 29^4, 31^4, 47^4$  
$7^8, 10^1, 10^1, 11^10, 23^10, 3^12$ |
| 7    | $137^2, 139^2, 141^2, 151^2, 167^2, 193$ |
| 8    | $157^2, 163^2, 173^2, 179^2, 191^2$ |
| 9    | $181^2, 197^2, 199^2$ |
| 10   | $193^2, 211^2, 223^4, 227^2, 239^2, 37^4, 41^4, 43^4, 53^4, 59^4$  
$71^4, 13^6, 17^6, 19^6, 29^6, 31^6, 47^6, 51^2, 7^12, 111^2, 23^12$  
$3^14, 5^14, 7^14, 11^14, 23^14, 5^16, 7^16, 11^16, 23^16, 31^8, 3^20$ |

$^+$ Lower estimates
3. The Main Theorem

Theorem 4.3 Let $\bar{G}$ be a maximal solvable primitive permutation group of rank $r$ and degree $d$. Then either

$$ r > 24^{1/3}d^{0.02747-1.5} $$  \hspace{2cm} (1) 

or $\bar{G}$ is permutation isomorphic to $S(p^m)$ wr $K$, where $K$ is a maximal transitive solvable subgroup of $S_k$, $d = p^{mk}$, and

$$ r > 24^{1/3}(1.188)^k-1.5 $$  \hspace{2cm} (2) 

Proof: Let $G$ be the stabilizer of $\bar{G}$ and suppose $G = H$ wr $K$, where $H$ is a maximal solvable primitive subgroup of $GL(m,p)$, $K$ is a maximal transitive solvable subgroup of $S_k$, $k > 1$, and $d = p^n$, $n = mk$ (by Theorem 1.1 and Theorem 3.3).

Suppose first that $H = T(p^m)$. Then by Theorem 4.2 $\bar{G}$ is permutation isomorphic to $S(p^m)$ wr $K$, and by Theorem 3.6 we have (2) (for $h = 2$, $a_h = 1.188$).

So now suppose $H \neq T(p^m)$. Let $h$ be the number of orbits of $H$, and suppose $k > 1$. Then by Theorem 3.6,

$$ r = (h)K \geq r(k,h) > 24^{1/3}a_h^{-k} - 1.5 $$

Thus it is enough to show that $d^{0.02747} < a_h^{-k}$. But $k = \log(p^m)(d)$, and thus $a_h^{-k} = d^{\ln(a_h)/\ln(p^m)}$. Thus it is enough to show that $0.02747 < \ln(a_h)/\ln(p^m)$, or
\[
p < \frac{1}{0.02747m}
\]

Now \( r(4, h) > h^4/24 \), and \( r(8, h) > h^8/168 > h^8/(24^{(8-1)/3}) \), and so
\[
a_h = \left[ \frac{r(d_h, h) + d_h}{24^{1/3}} \right]^{1/d_h} > h^{1/2} \quad \text{for all } h.
\]

By Theorem 2.3 we know that \( h > \frac{p^{m/2}}{12m} \) with the possible exceptions of \( p^m = 17^4, 19^4, 7^6, 5^8, 7^8, 11^8, 13^8, 7^9, 3^16, 5^16 \).

Thus for (3) to hold it is enough if
\[
p < \left[ \frac{p^{m/2}}{(24^{1/3}12m)} \right]^{1/(0.02747m)}
\]
or
\[
p < \frac{0.48253}{(24^{1/3}12m)^{1/m}}
\]

Now \( (24^{1/3}12m)^{1/m} \) decreases with \( m \) for \( m > 1 \), and thus if (4) is true for \( m = m_0 \), it is true for all \( m > m_0 \). So consider first \( m = 2 \). Then (4) is true for \( m = 2 \) if and only if \( p > 88.56 \), or \( p > p_2 \) for \( p_2 = 89 \). Similarly for each \( m < 20 \) there is some \( p_m \) with (4) true for all \( p > p_m \):

<table>
<thead>
<tr>
<th>( m )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>12</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_m )</td>
<td>89</td>
<td>29</td>
<td>13</td>
<td>11</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

This gives sixty-four more possible exceptions.

So let us now consider the seventy-four possible exceptions. Since \( a_h \) increases as \( h \) increases we need only show that for each possible exception \( p^m \) there is some \( h^* > h \) and some \( m^* > m \) such that \( p, m^*, h^* \) satisfy (3). Thus, using Table 8, we can eliminate exceptions as follows:
Thus all of the possible exceptions have been eliminated and the case \( k > 1 \) is complete.

So now suppose \( k = 1 \). Then \( G = H \wr K \cdot H \) since \( K = S_1 \), and so by Theorem 2.3 we have

\[
r > r^* (p, n)^+ 1 > p^{n/2} / 12n
\]

with ten possible exceptions. Thus we want

\[
p^{n/2} / 12n > 24 1/3, 0.02747n - 1.5
\]  

(5)

If \( n > 20 \) then (5) is true for all \( p > 2 \). For each \( n < 20 \) there exists some \( p_n \) such that for all \( p > p_n \) (5) is true, and these \( p_n \) can be summarized as follows:
<table>
<thead>
<tr>
<th>n</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>7</th>
<th>11</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>p_n</td>
<td>53</td>
<td>19</td>
<td>11</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

so we have forty-nine further possible exceptions.

So let us consider the fifty-nine possible exceptions. Since $24^{1/3}d^{0.02747} - 1.5$ increases with $d$, let $d_r$ be the greatest $d$ having $r > 24^{1/3}d^{0.02747} - 1.5$; i.e.

$$d_r = [(r+1.5)/24^{1/3}]^{1/0.02747}.$$ Then by computing $d_r$ and using Table IX we can eliminate possible exceptions as follows:

<table>
<thead>
<tr>
<th>r</th>
<th>$d_r$</th>
<th>Exceptions Eliminated</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1142.34</td>
<td>$2^2, 3^2, \ldots, 31^2, 2^3, 3^3, 5^3, 7^3, 2^4, 3^4, 5^4$</td>
</tr>
<tr>
<td>3</td>
<td>10740371</td>
<td>$3^2, \ldots, 53^2, 7^4, 11^4, \ldots, 31^4, 11^3, 13^3, 17^3, 19^3$</td>
</tr>
<tr>
<td>4</td>
<td>$1.59 \times 10^{10}$</td>
<td>$11^8, 13^8, 7^9, 2^{13}, 2^{14}, \ldots, 2^{19}, 3^{16}$</td>
</tr>
</tbody>
</table>

Thus all possible exceptions have been eliminated and the case $k = 1$ is complete.

This completes the proof of the theorem.

Theorem 4.4 For any $N > 0$ there exists a maximal solvable primitive permutation group $G$ of rank $r$ and degree $d > N$ such that $G$ has no initial semilinear component and $24^{1/3}d^{0.02747} - 1.5 < r < 24^{1/3}d^{0.02757} - 1.5$.
Proof: Let $H$ be the maximal solvable primitive non-semilinear subgroup of $GL(2,23)$ as constructed in §2.3. Then the number of nontrivial orbits of $H$ is $r^*(23,2,1) = 1$, and so $H$ has 2 orbits. Let $K_t$ be the group of degree $8 \cdot 4^{(t-1)}$ constructed in §3.6. Let $G = H \wr K_t$. Then $G$ is a maximal irreducible solvable subgroup of $GL(4^{(t+1)},23)$. Let $\mathcal{G}$ be the maximal solvable primitive permutation group with stabilizer $G$ as in Theorem 1.2. Then $\mathcal{G}$ has degree $n = 23^{4^{(t+1)}}$ and the rank $r$ of $\mathcal{G}$ is the number of orbits of $G$. Thus $r = (2)K_t$, and so for $m = \log_{23^2}(d) = 8 \cdot 4^{(t-1)}$,

$$r < 24^{1/3}a_h^m - 1.5 < 24^{1/3}(1.18872)^m - 1.5$$

and so

$$r < 24^{1/3}d \ln(1.18872)/\ln(23^2) - 1.5$$

$$< 24^{1/3}d^{0.02757} - 1.5$$

and this completes the proof of the theorem.

Finally note that any subgroup of a maximal primitive solvable permutation group $\mathcal{G}$ has rank greater than or equal to the rank of $\mathcal{G}$. Thus by reversing (2) in Theorem 4.3 we have the following:

**Theorem 4.5** Let $\mathcal{G}$ be a primitive solvable subgroup of $S_d$ with no semilinear component. Then if $r$ is the rank of $\mathcal{G}$, then we have $d < [(r+1.5)/24^{1/3}]^{1/0.02747}$. 
References


END
28H10H85
FIN